ANISOTROPIC QUATERNION CARNOT GROUPS: GEOMETRIC ANALYSIS AND GREEN'S FUNCTION

DER-CHEN CHANG, IRINA MARKINA

Abstract. We construct examples of 2-step Carnot groups related to quaternions and study their fine structure and geometric properties. This involves the Hamiltonian formalism, which is used to obtain explicit equations for geodesics and the computation of the number of geodesics joining two different points on these groups. We able to find the explicit lengths of geodesics. We present the fundamental solutions of the Heat and sub-Laplace equations for these anisotropic groups and obtain some estimates for them, which may be useful.

1. Introduction

This paper presents examples of Carnot groups and studies their fine structure, geometric properties and basic differential operators attached to them. A Carnot group is a connected and simply connected m-step nilpotent Lie group $\mathbb{G}$ whose Lie algebra $\mathcal{G}$ decomposes into the direct sum of vector subspaces $V_1 \oplus V_2 \oplus \ldots \oplus V_m$ satisfying the following relations:

$$[V_1, V_k] = V_{k+1}, \qquad 1 \le k < m, \qquad [V_1, V_m] = \{0\}.$$

The simplest examples of this group are Euclidean space $\mathbb{R}^n$, Heisenberg group $\mathbb{H}^n$ and $\mathbb{H}$(eisenberg)-type groups introduced by Kaplan [18]. The Carnot groups form a natural habitat for extensions of many of the objects studied in Euclidean space and find applications in the study of strongly pseudoconvex domains in complex analysis, semiclassical analysis of quantum mechanics, control theory, probability theory of degenerate diffusion process and others. The geometry of Carnot groups and differential operators related with them were studied extensively by many mathematicians, for instance, in [3, 4, 8, 9, 14, 16, 19, 20, 22].

We construct examples of 2-step Carnot groups, related to the multidimensional space of quaternion numbers. We will call these groups *anisotropic quaternion groups* and denote them by Q^n. In [12] the quaternion $\mathbb{H}$-type groups were studied. The results of [12] can be easily extended to the multidimensional quaternion space. The examples of the present paper contain the multidimensional quaternion $\mathbb{H}$-type groups as a particular case. We construct the Hamiltonian function associated with the sub-Laplacian generated by left invariant vector fields. Solving the Hamiltonian system of differential equations we give exact solutions that describe the geodesics on the group. We study geodesic connectivity between any two points of the group. It is known that every point of a Riemannian manifold is connected to every other point in a sufficiently small neighborhood by one single, unique geodesic. But in this case, there will be points arbitrarily near a point which are connected to this point by an infinite number of geodesics. Since we are working on a group, we may simply assume that the point is the origin $O = (0, 0)$. We prove the following results:

2000 *Mathematics Subject Classification.* 53C17, 53C22, 35H20.

Key words and phrases. Quaternions, Carnot-Carathéodory metric, nilpotent Lie groups, Hamiltonian formalism, Green function.

This work was supported by Projects FONDECYT (Chile) # 7050181, #1040333, and by grant of the University of Bergen.

(1). *If $P = (x, 0) = (x_1, \dots, x_n, 0)$ with $x \neq 0$, then there is only one geodesic connecting the origin O and the point P;*

(2). *If $P = (x_1, \dots, x_n, z)$ with $x_l \neq 0$, $l = 1, \dots, n$ and $z \neq 0$, then there are finitely many geodesics connecting the origin O and the point P;*

(3). *If $P = (x_1, \dots, x_n, z)$ with $x_l \neq 0$ for $l = 1, \dots, p-1$ and $x_l = 0$ for $l = p, \dots, n$, $z \neq 0$, then there are countably infinitely many geodesics connecting the point O and the point P;*

(4). *If $P = (0, z)$ with $z \neq 0$, then there are uncountably infinitely many geodesics connecting the point O and the point P.*

We will discuss basic properties of geodesics in sections 2 to 4. Then we will prove connectivity theorems in section 5 (see Theorems 5.1, 5.2, 5.4, and 5.6). Furthermore, parametric equations and arc lengths of all these geodesics are calculated explicitly.

We also consider complex geodesics and find a relation between the complex action function and the Carnot-Carathéodory metric. The complex action function allows us to deduce the transport equation and its solution: volume element. The fundamental solution of the Heat equation is given in terms of the complex action function and volume element. More precisely, the heat kernel at the origin is given by

$$P(y, w, t) = \frac{C}{t^{2n+3}} \int_{\mathbb{R}^3} e^{\frac{-f}{t}} V(\tau)\, d\tau,$$

where

$$f(y, w, \tau) = -i \sum_m \tau_m w_m + \sum_{l=1}^{n} \frac{|y_l|^2}{4} |\tau|_l \coth(|\tau|_l)$$

is the modified complex action and

$$V(\tau) = \prod_{l=1}^{n} \frac{|\tau|_l^2}{\sinh^2(|\tau|_l)}$$

is the volume element. (See Theorem 7.1). Integrating the fundamental solution of the Heat equation with respect to the time variable, we obtain the Green function for the sub-Laplacian (see Theorem 7.2):

$$G(x, z) = -\frac{2^{2n}(2\pi)^{2n+3}}{(2n+1)!} \int_{\mathbb{R}^3} \frac{V(\tau + i\varepsilon\tilde{z})}{f^{2n+2}(\tau + i\varepsilon\tilde{z})} d\tau.$$

The last section is devoted to some estimates of fundamental solutions, which may be useful.

Part of this paper was finished while the authors visited Universidad Técnica Federico Santa Marı́a, Valparaíso, Chile in December, 2005 under the grant Projects FONDECYT (Chile) # 7050181, #1040333. We would like to thank Professor Alexander Vasil'ev and the Departamento de Matemática of UTFSM for their invitation and the warm hospitality extended to them during their stay in Chile. We would also like to thank Professor Eric Grinberg for many inspired conversations on this project.

2. Definitions

A quaternion is a mathematical concept (re)introduced by William Rowan Hamilton from Ireland in 1843 [2]. (It has been said that *when Hamilton discovered the quaternions, they stayed discovered*). The idea captured the popular imagination for a time because it involved relatively simple calculations that abandon the commutative law, one of the basic rules of arithmetic. Specifically, a quaternion is a non-commutative extension of the complex numbers. As a vector space over the real numbers, the quaternions have dimension 4, whereas the complex numbers have dimension 2. While the complex numbers are obtained by adding the

element $\mathbf{i}$ to the real numbers which satisfies $\mathbf{i}^2 = -1$, the quaternions are obtained by adding the elements $\mathbf{i}, \mathbf{j}$, and $\mathbf{k}$ to the real numbers which satisfy the following relations

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1.$$

Unlike real or complex numbers, multiplication of quaternions is not commutative, e. g.,

$$\mathbf{ij} = -\mathbf{ji} = \mathbf{k}, \quad \mathbf{jk} = -\mathbf{kj} = \mathbf{i}, \quad \mathbf{ki} = -\mathbf{ik} = \mathbf{j}. \tag{2.1}$$

The quaternions are an example of a division ring, an algebraic structure similar to a field except for commutativity of multiplication. In particular, multiplication is still associative and every non-zero element has a unique inverse.

The quaternions can be written as a combination of a scalar and a vector in analogy with the complex numbers being representable as a sum of real and imaginary parts, $a \cdot 1 + b \cdot \mathbf{i}$. For a quaternion $h = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$ we call a scalar a the *real part* and the 3-dimensional vector $\mathbf{u} = b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$ is called the *imaginary part* of h and is a *pure quaternion.* In $\mathbb{R}^4$, the basis of quaternion numbers can be given by real matrices

$$\mathcal{U} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\mathcal{M}_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}, \qquad \mathcal{M}_2 = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \qquad \mathcal{M}_3 = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}.$$

We have

$$h = \begin{bmatrix} a & b & -d & -c \\ -b & a & -c & d \\ d & c & a & b \\ c & -d & -b & a \end{bmatrix} = a\mathcal{U} + b\mathcal{M}_1 + c\mathcal{M}_2 + d\mathcal{M}_3.$$

Similarly to complex numbers, vectors, and matrices, the addition of two quaternions is equivalent to summing up the coefficients. Set $h = a + \mathbf{u}$, and $q = t + x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = t + \mathbf{v}$. Then

$$h + q = (a + t) + (\mathbf{u} + \mathbf{v}) = (a + t) + (b + x)\mathbf{i} + (c + y)\mathbf{j} + (d + z)\mathbf{k}.$$

Addition satisfies all the commutation and association rules of real and complex numbers. The quaternion multiplication (the Grassmanian product) is defined by

$$hq = (at - \mathbf{u} \cdot \mathbf{v}) + (a\mathbf{v} + t\mathbf{u} + \mathbf{u} \times \mathbf{v}),$$

where $\mathbf{u} \cdot \mathbf{v}$ is the scalar product and $\mathbf{u} \times \mathbf{v}$ is the vector product of $\mathbf{u}$ and $\mathbf{v}$, both in $\mathbb{R}^3$. The multiplication is not commutative because of the non-commutative vector product. The non-commutativity of multiplication has some unexpected consequences, e.g., polynomial equations over the quaternions may have more distinct solutions then the degree of a polynomial. The equation $h^2 + 1 = 0$, for instance, has infinitely many quaternion solutions $h = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$ with $b^2 + c^2 + d^2 = 1$. The *conjugate* of a quaternion $h = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$, is defined as $h^* = a - b\mathbf{i} - c\mathbf{j} - d\mathbf{k}$ and the *absolute value* of h is defined as $|h| = \sqrt{hh^*} = \sqrt{a^2 + b^2 + c^2 + d^2}$.

Let us denote the space of quaternions by $\mathcal{H}$. We consider n-tuples of quaternions: $H = (h_1, h_2, \dots, h_n)$, $h_i \in \mathcal{H}$, $i = 1, \dots, n$. We may define addition between two of them in an obvious way:

$$H + Q = (h_1 + q_1, h_2 + q_2, \dots, h_n + q_n)$$

for $H = (h_1, \dots, h_n)$ and $Q = (q_1, \dots, q_n)$. Multiplication by scalar is defined as $\alpha H = (\alpha h_1, \alpha h_2, \dots, \alpha h_n)$, where α may be real or complex number. Therefore, the n-dimensional quaternions, $\mathcal{H}^n$ is a vector space. The norm is defined by

$$|H|_{\mathcal{H}^n} = \Big(\sum_{i=1}^{n} |h_i|^2\Big)^{1/2}.$$

In this article we will construct 2-step Carnot groups related to the multidimensional quaternion numbers. We will call these groups *anisotropic quaternion groups* Q^n. To explain precisely the idea of their construction we first introduce the notion of $\mathbb{H}$-type Carnot groups and then, making some modifications, we arrive at our main example.

$\mathbb{H}$-type homogeneous groups are simply connected 2-step Lie groups $\mathbb{G}$ whose algebras $\mathcal{G}$ are graded and carry an inner product such that

(i) $\mathcal{G}$ is the orthogonal direct sum of the generating subspace V_1 and the center V_2:

$$\mathcal{G} = V_1 \oplus V_2, \quad V_2 = [V_1, V_1], \quad [V_1, V_2] = 0,$$

(ii) the homomorphisms $J_{\mathcal{Z}} : V_1 \to V_1$, $\mathcal{Z} \in V_2$ defined by

$$\langle J_{\mathcal{Z}} X, X' \rangle = \langle \mathcal{Z}, [X, X'] \rangle, \quad X, X' \in V_1$$

satisfy the equation

$$J_{\mathcal{Z}}^2 = -|\mathcal{Z}|^2 I, \quad \mathcal{Z} \in V_2.$$

Here $\langle \cdot, \cdot \rangle$ is a positive definite non-degenerating quadratic form on $\mathcal{G}$, $[\cdot, \cdot]$ is a commutator and I is the identity. The group is generated from its algebra by exponentiation.

To construct the multidimensional quaternion $\mathbb{H}$-type group we take the space of quaternions $\mathcal{H}^n$ as V_1 and generate the center V_2. We consider the n-dimensional imaginary quaternions

$$\mathcal{Z}_1 = (a_1\mathbf{i}, \dots, a_1\mathbf{i}), \quad \mathcal{Z}_2 = (a_2\mathbf{j}, \dots, a_2\mathbf{j}), \quad \mathcal{Z}_3 = (a_3\mathbf{k}, \dots, a_3\mathbf{k})$$

with positive constants a_m. They have the following representation as real matrices $4n \times 4n$:

$$M_m = \begin{bmatrix} a_m\mathcal{M}_m & & 0 \\ & \ddots & \\ 0 & & a_m\mathcal{M}_m \end{bmatrix},$$

where there are n blocks on the diagonal of each matrix M_m, $m = 1, 2, 3$. The matrices M_m, $m = 1, 2, 3$, are the matrices associated to the homomorphisms $J_{\mathcal{Z}}$.

Now we extend the construction, introducing anisotropy to this very symmetric setting. We take an arbitrary n-dimensional imaginary quaternions

$$\mathcal{Z}_1 = (a_{11}\mathbf{i}, \dots, a_{1n}\mathbf{i}), \quad \mathcal{Z}_2 = (a_{21}\mathbf{j}, \dots, a_{2n}\mathbf{j}), \quad \mathcal{Z}_3 = (a_{31}\mathbf{k}, \dots, a_{3n}\mathbf{k}),$$

with $a_{ml} > 0$ for all $m = 1, 2, 3$ and $l = 1, \dots, n$. The representation as real matrices $4n \times 4n$ are following:

$$\mathbf{M_1} = \begin{bmatrix} a_{11}\mathcal{M}_1 & & 0 \\ & \ddots & \\ 0 & & a_{1n}\mathcal{M}_1 \end{bmatrix}, \qquad \mathbf{M_2} = \begin{bmatrix} a_{21}\mathcal{M}_2 & & 0 \\ & \ddots & \\ 0 & & a_{2n}\mathcal{M}_2 \end{bmatrix},$$

$$\mathbf{M_3} = \begin{bmatrix} a_{31}\mathcal{M}_3 & & 0 \\ & \ddots & \\ 0 & & a_{3n}\mathcal{M}_3 \end{bmatrix},$$

where there are n blocks on the diagonal of each matrix $\mathbf{M}_m$, $m = 1, 2, 3$. We construct and make the principal calculations for the anisotropic quaternion group Q^n with center V_2 of

topological dimension 3. For the other examples see Remark 2.1. The corresponding algebra $\mathcal{Q}^n$ is the two-step algebra $V_1\oplus V_2$. The topological dimensions of the group is $\dim Q^n = 4n+3$. The homogeneous dimension defined by the formula $\nu = \dim V_1 + 2\dim V_2$ plays an important role in analysis on homogeneous groups. We see that the homogeneous dimension is always greater than the topological dimension and in our case equals $\nu(Q^n) = 4n+6$. An isotropic case based on the one dimensional space of quaternions was studied in [12] and the anisotropic Heisenberg group was considered in [8].

We set the standard orthonormal systems $\{X_{kl}\} \in \mathbb{R}^{4n}$, $k = 1,2,3,4$, $l = 1,\dots,n$ and $\{Z_1, Z_2, Z_3\} \in \mathbb{R}^3$. We reserve the following indexes: $l = 1,\dots n$ denotes the coordinate index in $(h_1,\dots,h_n)$; $k = 1,2,3,4$ denotes the index of coordinates inside of each quaternion h_l and $m = 1,2,3$ is related to the coordinate index in V_2 or the index of the matrices $\mathbf{M}_m$. The matrices $\mathbf{M}_m$ transform the basis vectors in the following way

(2.2)
$$\begin{array}{llll}
\mathbf{M}_1X_{1l} = -a_{1l}X_{2l}, & \mathbf{M}_1X_{2l} = a_{1l}X_{1l}, & \mathbf{M}_1X_{3l} = -a_{1l}X_{4l}, & \mathbf{M}_1X_{4l} = a_{1l}X_{3l},\\
\mathbf{M}_2X_{1l} = a_{2l}X_{4l}, & \mathbf{M}_2X_{2l} = a_{2l}X_{3l}, & \mathbf{M}_2X_{3l} = -a_{2l}X_{2l}, & \mathbf{M}_2X_{4l} = -a_{2l}X_{1l},\\
\mathbf{M}_3X_{1l} = a_{3l}X_{3l}, & \mathbf{M}_3X_{2l} = -a_{3l}X_{4l}, & \mathbf{M}_3X_{3l} = -a_{3l}X_{1l}, & \mathbf{M}_3X_{4l} = a_{3l}X_{2l}.
\end{array}$$

We use the normal coordinates

$$q = (x,z) = (x_{11}, x_{21}, x_{31}, x_{41}, \dots, x_{1n}, x_{2n}, x_{3n}, x_{4n}, z_1, z_2, z_3)$$

for the elements

$$\exp\Big(\sum_{k,l} x_{kl}X_{kl} + \sum_m z_mZ_m\Big) \in Q^n.$$

The Baker-Campbell-Hausdorff formula

$$\exp(X+Z)\exp(X'+Z') = \exp(X+X', Z+Z' + \frac{1}{2}[X,X']),$$

for $X, X' \in V_1$, $Z, Z' \in V_2$ defines the multiplication law on Q^n. Precisely, we have

$$\begin{aligned}
L_q(q') &= L_{(x,z)}(x',z') = (x,z)\circ(x',z')\\
&= \big(x+x', z_1+z_1' + \frac{1}{2}(\mathbf{M}_1x,x'), z_2+z_2' + \frac{1}{2}(\mathbf{M}_2x,x'), z_3+z_3' + \frac{1}{2}(\mathbf{M}_3x,x')\big),
\end{aligned}$$

for $q = (x,z)$ and $q' = (x',z')$, where $(\mathbf{M}_mx, x')$ is the usual scalar product of the vector $\mathbf{M}_mx \in \mathbb{R}^{4n}$ by $x' \in \mathbb{R}^{4n}$. The multiplication "$\circ$" defines the left translation L_q of $q' = (x',z')$ by the element $q = (x,z)$ on the group Q^n.

We associate the Lie algebra $\mathcal{Q}^n$ of the group Q^n with the set of all left invariant vector fields of the tangent bundle TQ^n. The tangent bundle contains a natural subbundle $\mathcal{T}Q^n$ consisting of "horizontal" vectors. We call $\mathcal{T}Q^n$ the *horizontal bundle*. The horizontal bundle is spanned by the left-invariant vector fields $\widetilde{X}_{11}(x,z),\dots,\widetilde{X}_{4n}(x,z)$ with $\widetilde{X}_{kl}(0,0) = X_{kl}$, $k = 1,\dots,4$, $l = 1,\dots,n$, (see for example, [7], [8], [23]). In coordinates of the standard Euclidean basis $\frac{\partial}{\partial x_{kl}}$, $\frac{\partial}{\partial z_m}$, these vector fields are expressed as

$$\begin{aligned}
\widetilde{X}_{1l}(x,z) &= \frac{\partial}{\partial x_{1l}} + \frac{1}{2}\Big(+a_{1l}x_{2l}\frac{\partial}{\partial z_1} - a_{2l}x_{4l}\frac{\partial}{\partial z_2} - a_{3l}x_{3l}\frac{\partial}{\partial z_3}\Big),\\
\widetilde{X}_{2l}(x,z) &= \frac{\partial}{\partial x_{2l}} + \frac{1}{2}\Big(-a_{1l}x_{1l}\frac{\partial}{\partial z_1} - a_{2l}x_{3l}\frac{\partial}{\partial z_2} + a_{3l}x_{4l}\frac{\partial}{\partial z_3}\Big),\\
\widetilde{X}_{3l}(x,z) &= \frac{\partial}{\partial x_{3l}} + \frac{1}{2}\Big(+a_{1l}x_{4l}\frac{\partial}{\partial z_1} + a_{2l}x_{2l}\frac{\partial}{\partial z_2} + a_{3l}x_{1l}\frac{\partial}{\partial z_3}\Big),\\
\widetilde{X}_{4l}(x,z) &= \frac{\partial}{\partial x_{4l}} + \frac{1}{2}\Big(-a_{1l}x_{3l}\frac{\partial}{\partial z_1} + a_{2l}x_{1l}\frac{\partial}{\partial z_2} - a_{3l}x_{2l}\frac{\partial}{\partial z_3}\Big),
\end{aligned} \tag{2.3}$$

for $l=1,\dots,n$. The left invariant vector fields $\widetilde{Z}_m(x,z)$ with $\widetilde{Z}_m(0,0)=Z_m$, $m=1,2,3$, are simply the vector fields

$$\widetilde{Z}_m(x,z)=\frac{\partial}{\partial z_m}. \tag{2.4}$$

We write simply X_{kl} and Z_m instead of $\widetilde{X}_{kl}(x,z)$ and $\widetilde{Z}_m(x,z)$, if no confusion may arise. Note that if we fix m to be equal only to $1,2$ or 3 then the vector fields (2.3) are reduced to the anisotropic vector fields of anisotropic $\mathbb{H}^{2n}$ Heisenberg group and the group Q^n is isomorphic to anisotropic $\mathbb{H}^{2n}$ Heisenberg group, considered in [8]. We also use the notation $X_l=(X_{1l},\dots,X_{4l})$, $l=1,\dots,n$. We call the next vector

$$X=(X_{11},\dots,X_{4\,n})=\big(\nabla_x+\frac{1}{2}\sum_{m=1}^{3}\mathbf{M}_m x\frac{\partial}{\partial z_m}\big),$$

where $\nabla_x=\big(\frac{\partial}{\partial x_{11}},\dots,\frac{\partial}{\partial x_{4\,n}}\big)$ the *horizontal gradient*. Any vector field Y belonging to $\mathcal{T}Q^n$ is called the *horizontal vector field*. In particular, the horizontal gradient X is a horizontal vector field, that justifies the name "horizontal" gradient.

The commutation relations are as follows

$$[X_{1l},X_{2l}]=-a_{1l}Z_1,\quad [X_{1l},X_{3l}]=a_{3l}Z_3,\quad [X_{1l},X_{4l}]=a_{2l}Z_2,$$
$$[X_{2l},X_{3l}]=a_{2l}Z_2,\quad [X_{2l},X_{4l}]=-a_{3l}Z_3,\quad [X_{3l},X_{4l}]=-a_{1l}Z_1,$$

for any $l=1,\dots,n$.

A basis of one-forms dual to X_{kl},Z_m, is given by dx_{kl},ϑ_m, with

$$\vartheta_m=dz_m-\frac{1}{2}\big(\mathbf{M}_m x,dx\big). \tag{2.5}$$

Since the interior product $\vartheta_m(X_{kl})$ vanishes for all $m=1,2,3$, $k=1,\dots,4$, $l=1,\dots,n$ we have the product $\vartheta_m(Y)$ vanishing on all horizontal vector fields Y.

Remark 2.1. If we formally put $a_{1l}=0$, $l=1,\dots,n$, then we obtain another example of a quaternion anisotropic group with 2-dimensional center. The case $a_{1l}=a_{2l}=0$, $l=1,\dots,n$, corresponds to an anisotropic group with 1-dimensional center.

3. Horizontal curves and their geometric characteristics

Summarizing the results of the previous section we can say that Q^n is a space of $(4n+3)$ - tuples of real numbers $\mathbb{R}^{4n+3}$ where the commutative group operation "+" is replaced by the noncommutative law "$\circ$". Respectively, the left translation $L_x(x')=x+x'$ (that in the commutative case coincides with the right translation) is substituted by the left translation $L_q(q')=q\circ q'$. The corresponding Lie algebras are fundamentally different. The constant vector fields $X_i=\frac{\partial}{\partial x_i}$, $i=1,\dots,4n+3$, of Euclidean space are replaced by the vector fields (2.3) and (2.4). Moreover, since the group of vector fields (2.4) is completely generated by the group (2.3) by means of commutation relations, the geometry of the group Q^n is defined by the horizontal bundle $\mathcal{T}Q^n$. The velocity and the distance should respect the horizontal bundle $\mathcal{T}Q^n$. Since $[X_{k_il_j},X_{k_al_b}]\notin\mathcal{T}Q^n$ the horizontal bundle is not integrable, i.e., there is no surface locally tangent to it [1]. As we say, the geometry is defined by the horizontal bundle, so it is sufficient to define the Riemanian metric only on the horizontal bundle $\mathcal{T}Q^n$ of tangent bundle of Q^n. These kinds of manifolds have aquired the name subRiemanian manifolds. The definitions and basic notations of subRiemanian geometry can be found in, e.g., [24].

A continuous map $c(s):[0,1]\to Q^n$ is called a curve. We say that a curve $c(s)$ is *horizontal* if its tangent vector $\dot{c}(s)$ (if it exists) belongs to $\mathcal{T}Q^n$ at each point $c(s)$. In other words, there

are (measurable) functions $a_{kl}(s)$ such that $\dot{c}(s) = \sum_{k,l} a_{kl}(s)X_{kl}(c(s))$. We present some simple propositions that describe the geometry of the group Q^n.

Proposition 3.1. *A curve $c(s) = (x(s), z(s))$ is horizontal if and only if*

$$\dot{z}_m = \frac{1}{2}(\mathbf{M}_m x, \dot{x}), \quad m = 1, 2, 3, \tag{3.1}$$

where $\dot{x} = (\dot{x}_{11}, \dot{x}_{21}, \dot{x}_{31}, \dot{x}_{41}, \dots, \dot{x}_{1n}, \dot{x}_{2n}, \dot{x}_{3n}, \dot{x}_{4n})$.

Proof. We can write the tangent vector $\dot{c}(s)$ in the form

$$\begin{aligned}
\dot{c}(s) =& (\dot{x}(s), \dot{z}(s)) = \sum_{k,l} \dot{x}_{kl}(s)\frac{\partial}{\partial x_{kl}} + \sum_m \dot{z}_m(s)\frac{\partial}{\partial z_m} \\
=& \Big(\dot{x}(s), \nabla_x + \frac{1}{2}\sum_m \mathbf{M}_m x\frac{\partial}{\partial z_m}\Big) + \sum_m \Big(\dot{z}_m(s) - \frac{1}{2}\big(\mathbf{M}_m x, \dot{x}(s)\big)\Big)\frac{\partial}{\partial z_m} \\
=& \Big(\dot{x}(s), X(s)\Big) + \sum_m \Big(\dot{z}_m(s) - \frac{1}{2}\big(\mathbf{M}_m x, \dot{x}(s)\big)\Big)\frac{\partial}{\partial z_m}.
\end{aligned}$$

It is clear that $\dot{c}(s)$ is horizontal if and only if the coefficients in front of $\frac{\partial}{\partial z_m}$, $m = 1, 2, 3$, vanish. This proves Proposition 3.1. $\square$

Corollary 3.2. *If a curve $c(s) = (x(s), z(s))$ is horizontal, then*

$$\dot{c}(s) = \Big(\dot{x}(s), X\Big) = \sum_{k,l} \dot{x}_{kl}(s)X_{kl}.$$

It is easy to see the following statement.

Proposition 3.3. *The left translation L_q of a horizontal curve $c(s) = (x(s), z(s))$ is a horizontal curve $\tilde{c}(s) = L_q(c(s))$ with the velocity*

$$\dot{\tilde{c}}(s) = (L_q)_* \dot{c}(s) = \sum_{k,l} \dot{x}_{kl}(s)X_{kl}(\tilde{c}(s)) = \big(\dot{x}(s), X(\tilde{c}(s))\big). \tag{3.2}$$

Proposition 3.4. *The acceleration vector $\ddot{c}(s)$ of a horizontal curve $c(s)$ is horizontal.*

Proof. Let $c(s)$ be a horizontal curve. Then $\dot{c}(s) \in \mathcal{T}Q^n_{c(s)}$. Let us show that $\ddot{c}(s) \in \mathcal{T}Q^n_{c(s)}$. Differentiating equalities (3.1) of the horizontality condition and making use of $(\mathbf{M}_m x, x) = 0$ for $m = 1, 2, 3$ and any $x \in \mathbb{R}^{4n}$, we deduce that

$$\ddot{z}_m(s) = \frac{1}{2}\Big(\big(\mathbf{M}_m\dot{x}(s), \dot{x}(s)\big) + \big(\mathbf{M}_m x(s), \ddot{x}(s)\big)\Big) = \frac{1}{2}\big(\mathbf{M}_m x(s), \ddot{x}(s)\big)$$

for $m = 1, 2, 3$. Then the acceleration vector along $c(s)$ is

$$\begin{aligned}
\ddot{c}(s) =& \big(\ddot{x}(s), \nabla_x\big) + \big(\ddot{z}(s), \nabla_z\big) = \Big(\ddot{x}(s), \nabla_x + \frac{1}{2}\sum_m \mathbf{M}_m x(s)\frac{\partial}{\partial z_m}\Big) \\
&+ \sum_m \Big(\ddot{z}_m(s) - \frac{1}{2}\big(\ddot{x}(s), \mathbf{M}_m x(s)\big)\Big)\frac{\partial}{\partial z_m} = \big(\ddot{x}(s), X(c(s))\big).
\end{aligned}$$

This means that the vector $\ddot{c}(s)$ is horizontal. The proposition is proved. $\square$

4. Hamiltonian formalism

In this section we study the geometry of the anisotropic quaternion group Q^n making use of the Hamiltonian formalism. The geometry of the group is induced by the sub-Laplacian $\Delta_0 = \sum_{k,l} X_{kl}^2$. Operators of such type are studied, for instance, in [4, 10]. Since the vector fields X_{kl} satisfying the Chow's condition, by a theorem of Hörmander [17], the operator Δ_0 is hypoelliptic. Explicitly, the sub-Laplacian has the form:

$$\Delta_0 = \sum_{l=1}^{n}\sum_{k=1}^{4} X_{kl}^2 = \Big(\Delta_x + \frac{1}{4}\sum_{m=1}^{3}\Big(\sum_{l=1}^{n} a_{ml}^2|x_l|^2\Big)\frac{\partial^2}{\partial z_m^2} + \sum_{m=1}^{3}\big(\mathbf{M}_m x, \nabla_x\big)\frac{\partial}{\partial z_m}\Big),$$

where $\nabla_x = (\frac{\partial}{\partial x_{11}}, \dots, \frac{\partial}{\partial x_{4n}})$, $\Delta_x = \sum_{l=1}^{n}\sum_{k=1}^{4}\frac{\partial^2}{\partial x_{kl}^2}$, and $|x_l|^2 = \sum_{k=1}^{4} x_{kl}^2$. To present the Hamiltonian function we introduce the formal variables $\xi = (\xi_{11}, \dots, \xi_{4n})$ with $\xi_{kl} = \frac{\partial}{\partial x_{kl}}$ and $\theta = (\theta_1, \theta_2, \theta_3)$ with $\theta_m = \frac{\partial}{\partial z_m}$, $m = 1, 2, 3$. The associated with sub-Laplacian Δ_0 Hamiltonian function $H(\xi, \theta, x, z)$ is following

$$H(\xi,\theta,x,z) = |\xi|^2 + \frac{1}{4}\sum_{m=1}^{3}\Big(\sum_{l=1}^{n} a_{ml}^2|x_l|^2\Big)\theta_m^2 + \Big(\big(\sum_{m=1}^{3}\theta_m\mathbf{M}_m\big)x, \xi\Big), \tag{4.1}$$

where $|\xi|^2 = \sum_{l,k}\xi_{kl}^2$, and diagonal blocks of the matrix $\sum_{m=1}^{3}\theta_m\mathbf{M}_m$ are of the form

$$\begin{bmatrix} 0 & a_{1l}\theta_1 & -a_{3l}\theta_3 & -a_{2l}\theta_2 \\ -a_{1l}\theta_1 & 0 & -a_{2l}\theta_2 & a_{3l}\theta_3 \\ a_{3l}\theta_3 & a_{2l}\theta_2 & 0 & a_{1l}\theta_1 \\ a_{2l}\theta_2 & -a_{3l}\theta_3 & -a_{1l}\theta_1 & 0 \end{bmatrix}.$$

We use the following notation:

$$A_m^2 = \begin{bmatrix} a_{m1}^2\mathcal{U} & & 0 \\ & \ddots & \\ 0 & & a_{mn}^2\mathcal{U} \end{bmatrix},$$

$\mathbf{\Theta}^2 = \sum_{m=1}^{3}\theta_m^2 A_m^2$, and $\mathbf{M} = \sum_{m=1}^{3}\theta_m\mathbf{M}_m$. We also introduce some different metrics for convenience: $|\theta|_l^2 = \sum_{m=1}^{3}\theta_m^2 a_{ml}^2$, $|x|_B^2 = (B^2x, x) = (Bx, Bx)$, where B is a diagonal matrix. In this notation we get $\sum_{l=1}^{n} a_{ml}^2|x_l|^2 = |x|_{A_m}^2$. The Hamiltonian function takes a new form in this notation

$$H(\xi,\theta,x,z) = |\xi|^2 + \frac{1}{4}\sum_{m=1}^{3}|x|_{A_m}^2\theta_m^2 + \big(\mathbf{M}x, \xi\big) = |\xi|^2 + \frac{1}{4}(\mathbf{\Theta}^2 x, x) + \big(\mathbf{M}x, \xi\big), \tag{4.2}$$

and the corresponding Hamiltonian system obtains the form

$$\begin{cases} \dot{x} &= \frac{\partial H}{\partial \xi} = 2\xi + \mathbf{M}x \\ \dot{z}_m &= \frac{\partial H}{\partial \theta_m} = \frac{\theta_m}{2}|x|_{A_m}^2 + \big(\mathbf{M}_m x, \xi\big), \quad m = 1,2,3. \\ \dot{\xi} &= -\frac{\partial H}{\partial x} = -\frac{1}{2}\mathbf{\Theta}^2 x + \mathbf{M}\xi \\ \dot{\theta}_m &= -\frac{\partial H}{\partial z_m} = 0. \end{cases} \tag{4.3}$$

The solutions $\gamma(s) = (x(s), z(s), \xi(s), \theta(s))$ of the system (4.3) are called *bicharacteristics*.

Definition 4.1. Let $P_1(x_0, z_0)$, $P_2(x, z) \in Q^n$. A geodesic from P_1 to P_2 is the projection of a bicharacteristic $\gamma(s)$, $s \in [0, \tau]$, onto the (x, z)-space, that satisfies the boundary conditions

$$\big(x(0), z(0)\big) = (x_0, z_0), \qquad \big(x(\tau), z(\tau)\big) = (x, z).$$

The next properties of the matrices $\mathcal{M}_m$, $m=1,2,3$, are obvious:

$$\mathcal{M}_m^2=-\mathcal{U},\quad m=1,2,3,\ \text{where}\ \mathcal{U}\ \text{is the unit}\ (4\times 4)-\text{matrix}. \tag{4.4}$$

$$\begin{aligned}\mathcal{M}_1\mathcal{M}_2=-\mathcal{M}_2\mathcal{M}_1 &= \mathcal{M}_3,\quad \mathcal{M}_2\mathcal{M}_3=-\mathcal{M}_3\mathcal{M}_2=\mathcal{M}_1,\\ \mathcal{M}_3\mathcal{M}_1 &= -\mathcal{M}_1\mathcal{M}_3=\mathcal{M}_2.\end{aligned} \tag{4.5}$$

$$\mathcal{M}_m^{-1}=-\mathcal{M}_m,\ \text{where}\ \mathcal{M}_m^{-1}\ \text{is the inverse matrix of}\ \mathcal{M}_m,\ m=1,2,3. \tag{4.6}$$

$$\mathcal{M}_m^{T}=-\mathcal{M}_m,\ \text{where}\ \mathcal{M}_m^{T}\ \text{is the transposed matrix for}\ \mathcal{M}_m,\ m=1,2,3. \tag{4.7}$$

$$(\mathcal{M}_m x,x)=0,\ m=1,2,3,\ \text{for any}\ x\in\mathbb{R}^4. \tag{4.8}$$

As a corollary we obtain some useful formulas.

Proposition 4.2. *In the above-mentioned notations we have*

$$(\mathbf{M}_m x,\mathbf{M}x)=\theta_m\sum_{l=1}^{n}a_{ml}^2|x_l|^2=\theta_m|x|^2_{A_m}\quad \textit{for any}\quad j=1,2,3. \tag{4.9}$$

$$\mathbf{M}^2=-\sum_{m=1}^{3}\theta_m^2A_m^2=-\mathbf{\Theta}^2,\quad \mathbf{M}^3=-\mathbf{\Theta}^2\mathbf{M},\quad \mathbf{M}^4=\mathbf{\Theta}^4,\quad \mathbf{M}^5=\mathbf{\Theta}^4\mathbf{M}\ \dots \tag{4.10}$$

Proof.

$$\begin{aligned}(\mathbf{M}_m x,\mathbf{M}x) &= (\mathbf{M}_m x,\sum_{j=1}^{3}\theta_j\mathbf{M}_j x)=\sum_{j=1}^{3}\theta_j(\mathbf{M}_m x,\mathbf{M}_j x)\\ &= \sum_{j=1}^{3}\theta_j\Big((a_{j1}\mathcal{M}_jx_1,\dots,a_{jn}\mathcal{M}_jx_n)\ ,\ (a_{m1}\mathcal{M}_mx_1,\dots,a_{mn}\mathcal{M}_mx_n)\Big)\\ &= \sum_{j=1}^{3}\theta_j\sum_{l=1}^{n}a_{jl}a_{ml}(\mathcal{M}_jx_l,\mathcal{M}_mx_l)=\sum_{j=1}^{3}\theta_j\sum_{l=1}^{n}a_{jl}a_{ml}(-\mathcal{M}_m\mathcal{M}_jx_l,x_l)\\ &= \theta_m\sum_{l=1}^{n}a_{ml}^2|x_l|^2=\theta_m|x|^2_{A_m}\end{aligned}$$

by the properties (4.4), (4.5), (4.7), and (4.8) of matrix $\mathcal{M}_m$.

To prove (4.10) we note that $\mathbf{M}_m\mathbf{M}_j=-\mathbf{M}_j\mathbf{M}_m$ by (4.5) and $\mathbf{M}_m^2=-A_m^2$ by the property (4.4) for any $j,m=1,2,3$. Then

$$\mathbf{M}^2=\sum_{j=1}^{3}\sum_{m=1}^{3}\theta_j\theta_m\mathbf{M}_j\mathbf{M}_m=\sum_{m=1}^{3}\theta_m^2\mathbf{M}_m^2=-\sum_{m=1}^{3}\theta_m^2A_m^2=-\mathbf{\Theta}^2.$$

The rest is obvious. □

Lemma 4.3. *Any geodesic is a horizontal curve.*

Proof. Let $c(s)=(x(s),z(s))$ be a geodesic. The system (4.3) implies

$$\dot z_m=\frac{\theta_m}{2}|x|^2_{A_m}+\frac{1}{2}\big(\mathbf{M}_m x,2\xi\big)=\frac{\theta_m}{2}|x|^2_{A_m}+\frac{1}{2}\big(\mathbf{M}_m x,\dot x\big)+\frac{1}{2}\big(\mathbf{M}_m x,2\xi-\dot x\big). \tag{4.11}$$

Making use of the first line of the system (4.3), we write the last term of (4.11) as

$$\frac{1}{2}\big(\mathbf{M}_m x,2\xi-\dot x\big)=-\frac{1}{2}(\mathbf{M}_m x,\mathbf{M}x)=-\frac{\theta_m}{2}|x|^2_{A_m}. \tag{4.12}$$

Here we used the formula (4.9). Combining (4.11) and (4.12) we deduce

$$\dot z_m=\frac{\theta_m}{2}|x|^2_{A_m}+\big(\mathbf{M}_m x,\xi\big)=\frac{1}{2}\big(\mathbf{M}_m x,\dot x\big),\quad m=1,2,3. \tag{4.13}$$

Therefore, $c(s)$ is a horizontal curve by Proposition 3.1. □

Lemma 4.3 shows that the second equation of the system (4.3) is nothing more then the horizontality condition (3.1).

Let us try to solve the Hamiltonian system explicitly. The last equation in (4.3) shows that the function $H(\xi,\theta,x,z)$ does not depend on z. We obtain that θ_m are constants which can be used as Lagrangian multipliers. Multiplying the first line of system (4.3) by $\mathbf{M}$, we obtain

$$\mathbf{M}\dot x=2\mathbf{M}\xi-\mathbf{\Theta}^2x. \tag{4.14}$$

Expressing $\mathbf{M}\xi$ from (4.14) and substituting it in the equation for $\dot\xi$ from (4.3), we get

$$\dot\xi=\frac{\mathbf{M}\dot x}{2}. \tag{4.15}$$

We differentiate the first equation of (4.3) and substitute the $\dot\xi$ from (4.15). Finally, we deduce

$$\ddot x=2\dot\xi+\mathbf{M}\dot x=2\mathbf{M}\dot x. \tag{4.16}$$

Let us solve the equation (4.16). We substitute $y(s)=\dot x(s)$. The equation $\dot y(s)=2\mathbf{M}y(s)$ has a solution $y(s)=\exp(2s\mathbf{M})y(0)$. Therefore,

$$\dot x(s)=\exp(2s\mathbf{M})\dot x(0). \tag{4.17}$$

Let us discuss the properties of the matrix $\exp(2s\mathbf{M})$. For simplicity of notation we write $[B]_l$ for l-block of a block diagonal matrix B.

Lemma 4.4. *The exponent* $\exp\big(2s\mathbf{M}\big)$ *is an antisymmetric block matrix that commutes with* $\mathbf{M}$ *and which blocks can be written in the form:*

$$[\exp(2s\mathbf{M})]_l=\cos(2s|\theta|_l)\mathcal{U}+\frac{\sin(2s|\theta|_l)}{|\theta|_l}[\mathbf{M}]_l. \tag{4.18}$$

Proof. We observe that

$$\begin{aligned}[\exp\big(2s\mathbf{M})]_l=&\sum_{n=0}^{\infty}\frac{(2s)^n}{n!}[\mathbf{M}]_l^n=\mathbf{U}\sum_{k=0}^{\infty}\frac{(2s|\theta|_l)^{4k}}{(4k)!}+\frac{[\mathbf{M}]_l}{|\theta|_l}\sum_{k=0}^{\infty}\frac{(2s|\theta|_l)^{4k+1}}{(4k+1)!}\\&-\mathbf{U}\sum_{k=0}^{\infty}\frac{(2s|\theta|_l)^{4k+2}}{(4k+2)!}-\frac{[\mathbf{M}]_l}{|\theta|_l}\sum_{k=0}^{\infty}\frac{(2s|\theta|_l)^{4k+3}}{(4k+3)!}\end{aligned}$$

by (4.10). We conclude that the matrices $\mathbf{M}$ and $\exp\big(2s\mathbf{M}\big)$ commute. Note that

$$\sum_{k=0}^{\infty}\frac{(2s|\theta|_l)^{4k}}{(4k)!}-\sum_{k=0}^{\infty}\frac{(2s|\theta|_l)^{4k+2}}{(4k+2)!}=\cos(2s|\theta|_l)$$

and

$$\sum_{k=0}^{\infty}\frac{(2s|\theta|_l)^{4k+1}}{(4k+1)!}-\sum_{k=0}^{\infty}\frac{(2s|\theta|_l)^{4k+3}}{(4k+3)!}=\sin(2s|\theta|_l).$$

With this notation, $\exp\big(2s\mathbf{M}\big)$ is a block diagonal matrix with the blocks $[\exp\big(2s\mathbf{M}\big)]_l =$

$$=\begin{bmatrix} \cos(2s|\theta|_l) & \frac{a_{1l}\theta_1}{|\theta|_l}\sin(2s|\theta|_l) & -\frac{a_{3l}\theta_3}{|\theta|_l}\sin(2s|\theta|_l) & -\frac{a_{2l}\theta_2}{|\theta|_l}\sin(2s|\theta|_l) \\ -\frac{a_{1l}\theta_1}{|\theta|_l}\sin(2s|\theta|_l) & \cos(2s|\theta|_l) & -\frac{a_{2l}\theta_2}{|\theta|_l}\sin(2s|\theta|_l) & \frac{a_{3l}\theta_3}{|\theta|_l}\sin(2s|\theta|_l) \\ \frac{a_{3l}\theta_3}{|\theta|_l}\sin(2s|\theta|_l) & \frac{a_{2l}\theta_2}{|\theta|_l}\sin(2s|\theta|_l) & \cos(2s|\theta|_l) & \frac{a_{1l}\theta_1}{|\theta|_l}\sin(2s|\theta|_l) \\ \frac{a_{2l}\theta_2}{|\theta|_l}\sin(2s|\theta|_l) & -\frac{a_{3l}\theta_3}{|\theta|_l}\sin(2s|\theta|_l) & -\frac{a_{1l}\theta_1}{|\theta|_l}\sin(2s|\theta|_l) & \cos(2s|\theta|_l) \end{bmatrix}.$$

□

The group structure allows to restrict our considerations to the curves issuing from the origin. Hence, $x(0)=0$. The equation (4.17) has the form

$$\dot{x}_l(s)=\cos(2s|\theta|_l)\mathcal{U}\dot{x}_l(0)+\frac{\sin(2s|\theta|_l)}{|\theta|_l}[\mathbf{M}]_l\dot{x}_l(0),\quad l=1,\dots,n, \tag{4.19}$$

by (4.18). Integrating from 0 to s we get

$$x_l(s)=\frac{1-\cos(2s|\theta|_l)}{2|\theta|_l^2}[\mathbf{M}]_l\dot{x}_l(0)+\frac{\sin(2s|\theta|_l)}{2|\theta|_l}\mathcal{U}\dot{x}_l(0),\quad l=1,\dots,n. \tag{4.20}$$

Let us describe the z-components of a geodesic curve. If a curve is geodesic, then it is horizontal by Lemma 4.3, and we have

$$\begin{aligned}\dot{z}_m(s)=\frac{1}{2}\big(\mathbf{M}_m x(s),\dot{x}(s)\big)=\sum_{l=1}^{n}\Big(&\frac{\cos(2s|\theta|_l)(1-\cos(2s|\theta|_l))}{4|\theta|_l^2}\big([\mathbf{M}_m]_l[\mathbf{M}]_l\dot{x}_l(0),\dot{x}_l(0)\big)\\ &+\frac{\sin(2s|\theta|_l)(1-\cos(2s|\theta|_l))}{4|\theta|_l^3}\big([\mathbf{M}_m]_l[\mathbf{M}]_l\dot{x}_l(0),[\mathbf{M}]_l\dot{x}_l(0)\big)\\ &+\frac{\sin(2s|\theta|_l)\cos(2s|\theta|_l)}{4|\theta|_l}\big([\mathbf{M}_m]_l\dot{x}_l(0),\dot{x}_l(0)\big)\\ &+\frac{\sin^2(2s|\theta|_l)}{4|\theta|_l^2}\big([\mathbf{M}_m]_l\dot{x}_l(0),[\mathbf{M}]_l\dot{x}_l(0)\big)\Big),\end{aligned}$$

for $m=1,2,3$ by (4.19) and (4.20). The properties (4.8) and (4.9) imply

$$\big([\mathbf{M}_m]_l\dot{x}_l(0),\dot{x}_l(0)\big)=([\mathbf{M}_m]_l[\mathbf{M}]_l\dot{x}_l(0),[\mathbf{M}]_l\dot{x}_l(0)\big)=0,$$

and

$$\big([\mathbf{M}_m]_l[\mathbf{M}]_l\dot{x}_l(0),\dot{x}_l(0)\big)=-\big([\mathbf{M}_m]_l\dot{x}_l(0),[\mathbf{M}]_l\dot{x}_l(0)\big)=-\theta_m a_{ml}^2|\dot{x}_l(0)|^2.$$

Finally, we see that

$$\dot{z}_m(s)=\sum_{l=1}^{n}\Big(\frac{\theta_m a_{ml}^2|\dot{x}_l(0)|^2}{4|\theta|_l^2}\big(1-\cos(2s|\theta|_l)\big)\Big),\quad m=1,2,3. \tag{4.21}$$

Integrating equations (4.21), we get

$$z_m(s)=\sum_{l=1}^{n}\Big(\frac{\theta_m a_{ml}^2|\dot{x}_l(0)|^2}{4|\theta|_l^2}\big(s-\frac{\sin(2s|\theta|_l)}{2|\theta|_l}\big)\Big),\quad m=1,2,3. \tag{4.22}$$

Lemma 4.5. *Not all of horizontal curves are geodesics.*

Proof. To prove this proposition we present an example. The curve

$$c(s)=\Big(\frac{s^2}{2},s,\frac{s^2}{2},s,0,\dots,0,\frac{a_{11}s^3}{6},c_1,c_2\Big)$$

is horizontal with c_1,c_2 constant. Indeed,

$$\begin{aligned}
\dot z_1(s)=&\frac{a_{11}s^2}{2},\quad \frac{1}{2}\big(\mathbf{M}_1x,\dot x\big)=\frac{a_{11}}{2}(s^2-\frac{s^2}{2}+s^2-\frac{s^2}{2})=\frac{a_{11}s^2}{2},\\
\dot z_2(s)=&0,\quad \frac{1}{2}\big(\mathbf{M}_2x,\dot x\big)=\frac{a_{21}}{2}(-s^2-\frac{s^2}{2}+s^2+\frac{s^2}{2})=0,\\
\dot z_3(s)=&0,\quad \frac{1}{2}\big(\mathbf{M}_3x,\dot x\big)=\frac{a_{31}}{2}(-\frac{s^3}{2}+s+\frac{s^3}{2}-s)=0.
\end{aligned}$$

From the other hand, the curve $c(s)$ does not satisfy the system (4.16). The system (4.16) gets the form

$$\begin{cases}
1=2(a_{11}\theta_1-a_{31}\theta_3s-a_{21}\theta_2)\\
0=2(-a_{11}\theta_1s-a_{21}\theta_2s+a_{31}\theta_3)\\
1=2(a_{31}\theta_3s+a_{21}\theta_2+a_{11}\theta_1)\\
0=2(a_{21}\theta_2s-a_{31}\theta_3-a_{11}\theta_1s)
\end{cases}$$

for the curve $c(s)$. Summing up the first and the third equation, and then, the second and the forth ones, we write the latter system as follows

$$\begin{cases}
2=4a_{11}\theta_1\\
0=-4a_{11}\theta_1s\\
1=2(a_{31}\theta_3s+a_{21}\theta_2+a_{11}\theta_1)\\
0=2(a_{21}\theta_2s-a_{31}\theta_3-a_{11}\theta_1s).
\end{cases}$$

We see that the first and the second equations contradict each other. □

Lemma 4.6. *A curve c is a geodesic for the group Q^n if and only if*

(i) *$c(s)$ is a horizontal curve and*
(ii) *$c(s)$ satisfies $\ddot c(s)=2\mathbf{M}\dot c(s)$, $l=1,\dots,n$.*

Proof. If a curve is geodesic, then it is horizontal by Lemma 4.3. Proposition 3.4 implies that the vector $\ddot c$ is also horizontal: $\ddot c=\sum_{l=1}^n\ddot x_lX_l$. Since $\ddot x(s)=2\mathbf{M}\dot x$ by (4.16), we obtain the necessary result.

Let the curve $c(s)$ satisfy (i) and (ii) of Lemma 4.6. The horizontality condition (i) of Lemma 4.6 can be written in the form

$$\dot z_m=\frac{1}{2}\Big(\mathbf{M}_mx,\dot x\Big)=\frac{\theta_m}{2}|x|^2_{A_m}+\big(\mathbf{M}_mx,\xi\big)=\frac{\partial H}{\partial\theta_m},\quad m=1,2,3. \tag{4.23}$$

as in (4.13). We see that $c(s)$ satisfies the equations of the second line of (4.3). The condition (ii) of Lemma 4.6 admits the form $\ddot x(s)=2\mathbf{M}\dot x(s)$ in the coordinate functions. Define the following curve $\gamma(s)=(x(s),z(s),\xi(s),\theta)$ in the cotangent space, where

$$\xi=\frac{\dot x(s)}{2}-\frac{1}{2}\mathbf{M}x(s)\quad\text{with}\quad\theta=(\theta_1,\theta_2,\theta_3)\quad\text{constant.} \tag{4.24}$$

The relations (4.24) imply the equations of the first and the last lines of (4.3). Differentiating (4.24), we get

$$\dot\xi=\frac{\ddot x}{2}-\frac{1}{2}\mathbf{M}\dot x=\mathbf{M}\dot x-\frac{\mathbf{M}\dot x}{2}=\frac{1}{2}\mathbf{M}(2\xi+\mathbf{M}x)=\mathbf{M}\xi-\frac{1}{2}\mathbf{\Theta}^2x,$$

by the condition (ii) of Lemma 4.6, (4.24), and (4.10). Thus, $\gamma(s)$ satisfies the Hamilton system (4.3). Then, the projection onto the (x, z)-space, that coincides with $c(s)$, is a geodesic. □

5. Connectivity by geodesics.

Let us ask in the following question. Is it possible to join arbitrary two points of Q^n by a horizontal curve? A theorem by Chow [13] gives an affirmative answer. We present a direct proof and calculate the number of geodesics connecting the origin with different points. We need the next simple observation.

Proposition 5.1. *The kinetic energies $\mathcal{E} = \frac{1}{2}|\dot{x}|^2$, $\mathcal{E}_m = \frac{1}{2}|\dot{x}|^2_{A_m}$ are preserved along geodesics.*

Proof. In fact,

$$\frac{d\mathcal{E}_m}{ds} = \big(A_m\dot{x}, A_m\ddot{x}\big) = 2\big(A_m\dot{x}, \mathbf{M}A_m\dot{x}\big) = 0$$

by Lemma 4.6 and property (4.8) of the matrices $\mathcal{M}_m$. The same is for $\mathcal{E}$. □

5.2. Connectivity between $(0,0)$ and $(x,0)$, $x \neq 0$.

Theorem 5.1. *A smooth curve $c(s)$ is horizontal with constant z-coordinates z_1, z_2, z_3 if and only if $c(s) = (\alpha_{11}s, \dots, \alpha_{4\,n}s, z_1, z_2, z_3)$ with $\alpha_{kl} \in \mathbb{R}$ and $\sum_{l=1}^n \sum_{k=1}^4 \alpha_{kl}^2 \neq 0$. In other words, there is only one geodesic joining the origin with a point $(x, 0)$.*

Proof. Let $c(s)$ be a horizontal curve with constant z-coordinates z_1, z_2, z_3. Then $\dot{z}_m = 0$ and (4.21) implies

$$0 = \dot{z}_m = \theta_m \sum_{l=1}^n \frac{a_{ml}^2|\dot{x}_l(0)|^2}{2|\theta|_l^2}\sin^2\big(s|\theta|_l\big), \quad m = 1,2,3.$$

We define by continuity $\frac{\sin^2\big(s|\theta|_l\big)}{|\theta|_l^2} = s^2$ at $|\theta|_l = 0$. Since the sum $\sum_{l=1}^n \frac{a_{ml}^2|\dot{x}_l(0)|^2}{2|\theta|_l^2}\sin^2\big(s|\theta|_l\big)$ is not identically zero we deduce, that $\theta_m = 0$ for all $m = 1,2,3$. The Hamiltonian system (4.3) is reduced to the next one

$$\begin{cases} \dot{x} & = 2\xi \\ 0 & = \big(\mathbf{M}_m x, \xi\big), \quad m = 1,2,3 \\ \dot{\xi} & = 0 \\ \theta_m & = 0. \end{cases}$$

We see that ξ is a constant vector. Taking into account that $x(0) = 0$, we get $x(s) = (\alpha_{11}s, \dots, \alpha_{4\,n}s)$ with $\alpha_{kl} = 2\xi_{kl}$. This proves the statement.

Now, let us assume that $c(s) = (\alpha_{11}s, \dots, \alpha_{4\,n}s, z_1, z_2, z_3)$ with constant z-components. Set $\alpha s = (\alpha_{11}s, \dots, \alpha_{4\,n}s)$. Recall, that $(\mathbf{M}_m\alpha, \alpha) = 0$ for any vector $\alpha = (\alpha_{11}, \dots, \alpha_{4\,n})$ and $m = 1,2,3$. Then,

$$\dot{z}_m = 0 = \frac{1}{2}(\mathbf{M}_m(\alpha s), \dot{(\alpha s)}) = \frac{s}{2}(\mathbf{M}_m\alpha, \alpha), \qquad m = 1,2,3.$$

The horizontal condition (3.1) holds for all three z-components. □

5.3. **Connectivity between $(0,0)$ and $(0,z)$, $z \neq 0$.** We need to solve the equation (4.16) with the boundary conditions

$$x(0)=x(1)=z(0)=0, \qquad z(1)=z.$$

We also need to know the initial velocity $\dot{x}(0)$ since we do not have enough information about the behavior of x-coordinates. In the following theorem we use the notations $\mathbf{n}=(n_1,\dots,n_n)$, $n_l \in \mathbb{N}$, for $l=1,\dots,n$, and

$$N=\begin{bmatrix} \frac{1}{\pi n_1}\mathcal{U} & & 0 \\ & \ddots & \\ 0 & & \frac{1}{\pi n_n}\mathcal{U} \end{bmatrix}. \tag{5.1}$$

Theorem 5.2. *There are infinitely many geodesics joining the origin with a point $(0,z)$. The corresponding equations for each $\mathbf{n}=(n_1,\dots,n_n)$ are*

$$x_l^{(\mathbf{n})}(s)=2\frac{1-\cos(2s\pi n_l)}{(\pi n_l)^2}[\mathbf{Z}]_l\dot{x}_l(0)+\frac{\sin(2s\pi n_l)}{2\pi n_l}\mathcal{U}\dot{x}_l(0), \quad l=1,\dots,n, \tag{5.2}$$

where $\mathbf{Z}$ is a block diagonal matrix with the blocks

$$[\mathbf{Z}]_l=\begin{bmatrix} 0 & \frac{z_1a_{1l}}{|\dot{x}(0)|^2_{NA_1}} & -\frac{z_3a_{3l}}{|\dot{x}(0)|^2_{NA_3}} & -\frac{z_2a_{2l}}{|\dot{x}(0)|^2_{NA_2}} \\ -\frac{z_1a_{1l}}{|\dot{x}(0)|^2_{NA_1}} & 0 & -\frac{z_2a_{2l}}{|\dot{x}(0)|^2_{NA_2}} & \frac{z_3a_{3l}}{|\dot{x}(0)|^2_{NA_3}} \\ \frac{z_3a_{3l}}{|\dot{x}(0)|^2_{NA_3}} & \frac{z_2a_{2l}}{|\dot{x}(0)|^2_{NA_2}} & 0 & \frac{z_1a_{1l}}{|\dot{x}(0)|^2_{NA_1}} \\ \frac{z_2a_{2l}}{|\dot{x}(0)|^2_{NA_2}} & -\frac{z_3a_{3l}}{|\dot{x}(0)|^2_{NA_3}} & -\frac{z_1a_{1l}}{|\dot{x}(0)|^2_{NA_1}} & 0 \end{bmatrix}, \tag{5.3}$$

and

$$z_m^{(\mathbf{n})}(s)=\frac{z_m}{|\dot{x}(0)|^2_{NA_m}}\sum_{l=1}^n\frac{a_{ml}^2|\dot{x}(0)|^2}{|\pi n_l|^2}\Big(s-\frac{\sin(2s\pi n_l)}{2\pi n_l}\Big), \qquad m=1,2,3. \tag{5.4}$$

The lengths of corresponding geodesics are

$$l_{\mathbf{n}}^2=16\sum_{m=1}^3\frac{z_m^2(1)}{\sum_{l=1}^n\frac{a_{ml}^2|\dot{x}_l(0)|^2}{(\pi n_l)^2}}=16\sum_{m=1}^3\frac{z_m^2(1)}{|\dot{x}_l(0)|^2_{NA_m}}.$$

Proof. Substituting $s=1$ in (4.20), we calculate

$$\begin{aligned} 0 &= |x(1)|^2=\sum_{l=1}^n|x_l(1)|^2=\sum_{l=1}^n\Big(\frac{(\cos 2|\theta|_l-1)^2}{4|\theta|_l^4}\big([\mathbf{M}]_l\dot{x}_l(0),[\mathbf{M}]_l\dot{x}_l(0)\big)+\frac{\sin^2 2|\theta|_l}{4|\theta|_l^2}|\dot{x}_l(0)|^2\Big) \\ &= \sum_{l=1}^n\frac{\sin^2|\theta|_l}{|\theta|_l^2}|\dot{x}_l(0)|^2. \end{aligned}$$

Since the kinetic energy $\mathcal{E}=\frac{|\dot{x}(0)|^2}{2}$ does not vanish, there are indexes l, such that $|\dot{x}_l(0)|^2\neq 0$. We deduce in this case that

$$|\theta|_l=\sqrt{\theta_1^2a_{1l}^2+\theta_2^2a_{2l}^2+\theta_3^2a_{3l}^2}=\pi n_l, \qquad n_l\in\mathbb{N}.$$

If $|\dot{x}_l(0)|$=0, then the corresponding $|\theta|_l$ are arbitrary. Equalities (4.22) give for $s=1$

$$z_m(1)=\theta_m\sum_{l=1}^n\frac{a_{ml}^2|\dot{x}_l(0)|^2}{4(\pi n_l)^2}=\frac{\theta_m}{4}|\dot{x}(0)|^2_{NA_m}, \qquad m=1,2,3. \tag{5.5}$$

We find the unknown constants $\theta_m = \frac{4z_m(1)}{|\dot{x}(0)|^2_{NA_m}}$. Substituting θ_m in (4.20), (4.22), we obtain the equations (5.2) and (5.4) for geodesics.

To calculate the length of geodesics, we observe that $\sum_{m=1}^3 \theta_m^2 a_{ml}^2 = \pi^2 n_l^2$ and deduce

$$\sum_{m=1}^{3} z_m(1)\theta_m = \frac{1}{4}\sum_{m=1}^{3}\sum_{l=1}^{n}\frac{\theta_m^2 a_{ml}^2|\dot{x}_l(0)|^2}{(\pi n_l)^2} = \frac{1}{4}\sum_{l=1}^{n}\frac{|\dot{x}_l(0)|^2}{(\pi n_l)^2}\sum_{m=1}^{3}\theta_m^2 a_{ml}^2 = \frac{1}{4}|\dot{x}(0)|^2 = \frac{\mathcal{E}}{2}$$

from (5.5). The lengths of geodesics are

$$\begin{aligned} l_{\mathbf{n}}^2 &= \Big(\int_0^1 |\dot{x}(s)|\,ds\Big)^2 = |\dot{x}(0)|^2 = 4\sum_{m=1}^{3} z_m\theta_m = 16\sum_{m=1}^{3}\frac{z_m^2(1)}{\sum_{l=1}^{n}\frac{a_{ml}^2|\dot{x}_l(0)|^2}{(\pi n_l)^2}} \\ &= 16\sum_{m=1}^{3}\frac{z_m^2(1)}{|\dot{x}(0)|^2_{NA_m}}, \qquad \mathbf{n} = (n_1,\dots,n_n), \quad n_l \in \mathbb{N}. \end{aligned} \tag{5.6}$$

□

Remark 5.3. Let us discuss the cardinality of the set of geodesics. In the general case in Theorem 5.2, when a_{ml} are different we obtain the countably many geodesics connecting the origin with a point $P = (0, z)$. If the multiindex $\mathbf{n} = (n_1, \dots, n_n)$ increases, then the geodesic rotates more frequently around straight line, connecting origin with $P = (0, z)$ approaching to the line and in the limit we obtain a limit curve of Hausdorff dimension 2. We present a graphic of three geodesic curves (5.2),(5.4) with the initial velocity $\dot{x}(0) = (\dot{x}_{11}, 0, 0, 0, \dot{x}_{21}, 0, 0, 0,\ 0, \dots, 0)$ and the end point $P = (0, z_1, 0, 0)$, where $\dot{x}_{11}, \dot{x}_{21}, z_1$ are different from zero. The corresponding multiindexes are $\mathbf{n} = 1$, $\mathbf{n} = 2$ and $\mathbf{n} = 5$.

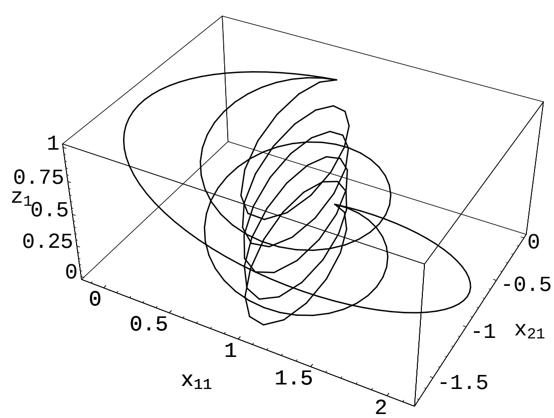

FIGURE 1. The graphs of geodesics where the vertical axis represents z coordinate and horizontal exes are x_{11} and x_{21}

The projection of geodesics into the horizontal subspace belongs to an ellipsoid passing through the origin.

If, in particular, $a_{1l} = a_{2l} = a_{3l} = a_l$ then rearranging the indexes, we can assume that $a_1 < a_2 < \ldots < a_p = a_{p+1} = \ldots = a_n$. Applying the rotation to the geodesics in the subspace $(0, \dots, 0, x_p, x_{p+1}, \dots, x_n, 0, 0, 0)$, we get uncountably many geodesics. In this case we have the

following estimate of their lengths:

$$l_{\mathbf{n}}^2 = \frac{16|z(1)|^2}{\sum_{l=1}^n \frac{a_l^2|\dot{x}_l(0)|^2}{(\pi n_l)^2}}.$$

If $a_{m1} = a_{m2} = \ldots = a_{mn} = a_m$, then the multiindex $\mathbf{n}$ reduces to the index $k \in \mathbb{N}$. The equation (5.6) implies

$$l_k^2 = |\dot{x}(0)|^2 = 4\pi k \sum_{m=1}^3 \frac{z_m^2(1)}{a_m^2}, \qquad k \in \mathbb{N}.$$

Let U be a neighborhood of the origin O. From Theorem 5.2, we know that no matter how small U is, we can always find points in U which are connected to O by an infinite number of geodesics. This is totally different from the Riemannian geometry. It is known that every point of a Riemannian manifold is connected to every other point in a sufficiently small neighborhood by one single, unique geodesic.

5.4. **Connectivity between** $(0,0)$ **and** (x,z)**,** $x \neq 0$**,** $z \neq 0$**.** Now, we will look for a solution of the equation (4.16) with the boundary conditions

$$x(0) = 0, \quad z(0) = 0, \qquad x(1) = x, \quad z(1) = z.$$

Let us make some preliminary calculations. We obtain

$$|\dot{x}_l(0)|^2 = \frac{|\theta|_l^2}{\sin^2|\theta|_l}|x_l(1)|^2, \quad l = 1, \ldots, n, \tag{5.7}$$

from (4.20) for $s = 1$. Putting $s = 1$ in (4.22) and making use of (5.7) we obtain

$$z_m(1) = \frac{\theta_m}{4}\sum_{l=1}^n \frac{a_{ml}^2|x_l(1)|^2}{|\theta|_l}\mu(|\theta|_l), \quad m = 1, 2, 3. \tag{5.8}$$

where $\mu(|\theta|_l) = \frac{|\theta|_l}{\sin^2(|\theta|_l)} - \cot(|\theta|_l)$. The function $\mu(\theta)$, introduced by Gaveau in [16], was first studied in detailed by Beals, Gaveau, Greiner in [5, 6, 7]. By the following lemma, one finds some basic properties of the function μ.

Lemma 5.5. *The function $\mu(\theta) = \frac{\theta}{\sin^2\theta} - \cot\theta$ is an increasing diffeomorphism of the interval $(-\pi, \pi)$ onto $\mathbb{R}$. On each interval $(m\pi, (m+1)\pi)$, $m = 1, 2, \ldots$, the function μ has a unique critical point c_m. On this interval the function μ strictly decreases from $+\infty$ to $\mu(c_m)$, and then, strictly increases from $\mu(c_m)$ to $+\infty$. Moreover,*

$$\mu(c_m) + \pi < \mu(c_{m+1}), \qquad m = 1, 2, \ldots.$$

The graph of $\mu(\theta)$ is given in Figure 2.

Theorem 5.4. *Given a point $P(x,z)$ with $x_l \neq 0$, $l = 1, \ldots, n$, $z \neq 0$, there are finitely many geodesics joining the point $O(0,0)$ with a point P. Let $\vartheta_{(1)} = (|\theta_1|_1, \ldots, |\theta_1|_n), \ldots, \vartheta_{(N)} = (|\theta_N|_1, \ldots, |\theta_N|_n)$ be solutions of the system*

$$\sum_{m=1}^3 \frac{16 z_m^2(1) a_{ml}^2}{\left(\sum_{r=1}^n \frac{a_{mr}^2|x_r(1)|^2\mu(|\theta|_r)}{|\theta|_r}\right)^2} = |\theta|_l^2, \quad l = 1, \ldots, n. \tag{5.9}$$

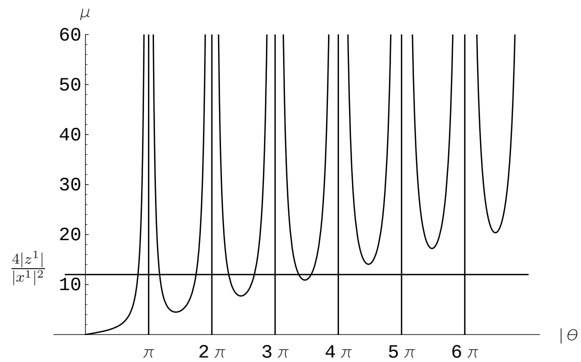

FIGURE 2. The graph of $\mu(\theta)$ and solutions of the equation $\mu(|\theta|) = \frac{4|z(1)|}{|x(1)|^2}$

We fix one of the solution $\vartheta = (|\theta|_1, \dots, |\theta|_n)$. Then the equations of the geodesics are

(5.10)
$$\begin{aligned} x_l^{(\mathbf{n})}(s) =& \big(4\cot(|\theta|_l)\sin^2(s|\theta|_l) - 2\sin(2s|\theta|_l)\big)\frac{[\mathbf{Z}]_l}{|\theta|_l}x_l(1) \\ &+ \big(\frac{1}{2}\cot|\theta|_l\sin(2s|\theta|_l) + \sin^2(s|\theta|_l)\big)\mathcal{U}x_l(1), \quad l = 1,\dots,n, \quad \mathbf{n} = 1,2,\dots,N, \end{aligned}$$

$$z_m^{(\mathbf{n})}(s) = z_m(1)\frac{\sum_{l=1}^n \frac{a_{ml}^2|x(1)|_l^2}{\sin^2(|\theta|_l)}\big(s - \frac{\sin(2s|\theta|_l}{2|\theta|_l})\big)}{\sum_{l=1}^n \frac{a_{ml}^2|x(1)|_l^2}{|\theta|_l}\mu(|\theta|_l)}, \quad m = 1,2,3, \quad \mathbf{n} = 1,2,\dots,N, \tag{5.11}$$

where $\mathbf{Z}$ is a block diagonal matrix with blocks (5.16)*. The lengths of these geodesics are*

$$l_{\mathbf{n}}^2 = \sum_{l=1}^n \frac{|\theta|_l^2|x_l(1)|^2}{\sin^2(|\theta|_l)} = 16\sum_{m=1}^3 \frac{z_m^2(1)}{\sum_{l=1}^n \frac{a_{ml}^2|x_l(1)|^2}{|\theta|_l}\mu(|\theta|_l)} + \sum_{l=1}^n |x_l(1)|^2|\theta|_l\cot(|\theta|_l). \tag{5.12}$$

Proof. We have

$$\theta_m = \frac{4z_m(1)}{\sum_{l=1}^n \frac{a_{ml}^2|x_l(1)|^2}{|\theta|_l}\mu(|\theta|_l)}, \quad m = 1,2,3, \tag{5.13}$$

from (5.8). Then

$$|\theta|_l^2 = \sum_{m=1}^3 \theta_m^2 a_{ml}^2 = \sum_{m=1}^3 \frac{16z_m^2a_{ml}^2}{\Big(\sum_{r=1}^n \frac{a_{mr}^2|x_r(1)|^2\mu(|\theta|_r)}{|\theta|_r}\Big)^2}, \quad l = 1,\dots,n, \tag{5.14}$$

that prove (5.9).

Let us fix one of the solutions of the equation (5.9) $\vartheta = (|\theta|_1, \dots, |\theta|_n)$ for a given point $P(x,z)$. Putting (5.7) and (5.13) into (4.22), we obtain (5.11).

Setting $s = 1$ in (4.20), we find $\dot{x}_l^{(\mathbf{n})}(0)$ for $\vartheta = (|\theta|_1, \dots, |\theta|_n)$:

$$\dot{x}_l^{(\mathbf{n})}(0) = 2|\theta|_l\Big[\sin(2|\theta|_l)\mathcal{U} + \big(1 - \cos(2|\theta|_l)\big)\frac{[\mathbf{M}]_l}{|\theta|_l}\Big]^{-1}x_l(1) = \Big[\big(|\theta|_l\cot|\theta|_l\big)\mathcal{U} - [\mathbf{M}]_l\Big]x_l(1).$$

This and (4.20) imply

$$\begin{aligned} x_l^{(\mathbf{n})}(s) &= \frac{1}{2}\Big[\big(2\cot|\theta|_l\sin^2(s|\theta|_l)-\sin(2s|\theta|_l)\big)\frac{[\mathbf{M}]_l}{|\theta|_l} \\ &+ \big(\cot|\theta|_l\sin(2s|\theta|_l)+2\sin^2(s|\theta|_l)\big)\mathcal{U}\Big]x_l(1), \qquad l=1,\dots,n. \end{aligned} \tag{5.15}$$

Taking into account (5.13), we deduce that each block $[\mathbf{M}]_l$ of the matrix $\mathbf{M}$ takes the form $4[\mathbf{Z}]_l$ with block $[\mathbf{Z}]_l$ written as

$$\begin{bmatrix} 0 & \frac{z_1(1)a_{1l}}{\sum_{l=1}^n \frac{a_{1l}^2|x_l|^2}{|\theta|_l}\mu(|\theta|_l)} & -\frac{z_3(1)a_{3l}}{\sum_{l=1}^n \frac{a_{3l}^2|x_l|^2}{|\theta|_l}\mu(|\theta|_l)} & -\frac{z_2(1)a_{2l}}{\sum_{l=1}^n \frac{a_{2l}^2|x_l|^2}{|\theta|_l}\mu(|\theta|_l)} \\ -\frac{z_1(1)a_{1l}}{\sum_{l=1}^n \frac{a_{1l}^2|x_l|^2}{|\theta|_l}\mu(|\theta|_l)} & 0 & -\frac{z_2(1)a_{2l}}{\sum_{l=1}^n \frac{a_{2l}^2|x_l|^2}{|\theta|_l}\mu(|\theta|_l)} & \frac{z_3(1)a_{3l}}{\sum_{l=1}^n \frac{a_{3l}^2|x_l|^2}{|\theta|_l}\mu(|\theta|_l)} \\ \frac{z_3(1)a_{3l}}{\sum_{l=1}^n \frac{a_{3l}^2|x_l|^2}{|\theta|_l}\mu(|\theta|_l)} & \frac{z_2(1)a_{2l}}{\sum_{l=1}^n \frac{a_{2l}^2|x_l|^2}{|\theta|_l}\mu(|\theta|_l)} & 0 & \frac{z_1(1)a_{1l}}{\sum_{l=1}^n \frac{a_{1l}^2|x_l|^2}{|\theta|_l}\mu(|\theta|_l)} \\ \frac{z_2(1)a_{2l}}{\sum_{l=1}^n \frac{a_{2l}^2|x_l|^2}{|\theta|_l}\mu(|\theta|_l)} & -\frac{z_3(1)a_{3l}}{\sum_{l=1}^n \frac{a_{3l}^2|x_l|^2}{|\theta|_l}\mu(|\theta|_l)} & -\frac{z_1(1)a_{1l}}{\sum_{l=1}^n \frac{a_{1l}^2|x_l|^2}{|\theta|_l}\mu(|\theta|_l)} & 0 \end{bmatrix}, \tag{5.16}$$

Finally, (5.16) and (5.15) give (5.10).

To obtain the length of the geodesics, we make the following calculations.

$$\begin{aligned} \sum_{m=1}^3 z_m(1)\theta_m &= \frac{1}{4}\sum_{l=1}^n \frac{|x_l(1)|^2\mu(|\theta|_l)}{|\theta|_l}\sum_{m=1}^3\theta_m^2a_{ml}^2 = \frac{1}{4}\sum_{l=1}^n |x_l(1)|^2|\theta|_l\mu(|\theta|_l) \\ &= \frac{1}{4}\sum_{l=1}^n \frac{|x_l(1)|^2|\theta|_l^2}{\sin^2(|\theta|_l)} - \frac{1}{4}\sum_{l=1}^n |x_l(1)|^2|\theta|_l\cot(|\theta|_l) \\ &= \frac{|\dot{x}(0)|^2}{4} - \frac{1}{4}\sum_{l=1}^n |x_l|^2|\theta|_l\cot(|\theta|_l). \end{aligned} \tag{5.17}$$

From the other hand (5.13) implies

$$\sum_{m=1}^3 z_m(1)\theta_m = 4\sum_{m=1}^3 \frac{z_m^2(1)}{\sum_{l=1}^n \frac{a_{ml}^2|x_l(1)|^2}{|\theta|_l}\mu(|\theta|_l)}. \tag{5.18}$$

The formula (5.12) follows from (5.17) and (5.18). □

Remark 5.5. Let us consider the particular case $a_{1l}=a_{2l}=a_{3l}=a_l>0$. We have

$$4|z| = \sum_{l=1}^n |a_l||x_l(1)|^2\mu(a_l|\theta|) \tag{5.19}$$

from (5.14). Here $|\theta|^2=\theta_1^2+\theta_2^2+\theta_3^2$. We denote by $|\theta|_1,\dots,|\theta|_N$ the solutions of (5.19) and let $|\theta|$ one of the solutions. Then (5.13) implies

$$\theta_m = \frac{4z_m(1)|\theta|}{\sum_{l=1}^n a_l|x_l|^2\mu(a_l|\theta|)} = \frac{z_m}{|z|}|\theta|.$$

To obtain the formula for the length of geodesics, we write (5.19) as

$$4|z| = \frac{1}{|\theta|}\sum_{l=1}^n \frac{|x_l|^2|\theta|_l^2}{\sin^2(|\theta|_l)} - \sum_{l=1}^n |a_l||x_l|^2\cot(|\theta|_l) = \frac{l_{\mathbf{n}}^2}{|\theta|} - \sum_{l=1}^n |a_l||x_l|^2\cot(|\theta|_l)$$

and get

$$l_{\mathbf{n}}^2 = |\theta|\big(4|z(1)| + \sum_{l=1}^{n} a_l|x_l(1)|^2\cot(a_l|\theta|)\big).$$

Simplifying more the situation and supposing that $a_l = a > 0$ for all $l = 1,\dots,n$, we get that $|\theta|_l = a|\theta|$. This implies that $|\theta|$ is a solution of the equation (see Figure 2)

$$\mu(a|\theta|) = \frac{4|z(1)|}{a|x(1)|^2}. \tag{5.20}$$

In this case to calculate the length of geodesics joining $(0,0)$ to (x,z), $x_l \neq 0$, $l = 1,\dots,n$, we use the homogeneous norm $|(x,z)|^2 = |x|^2 + 4|z|$. It gives for a solution $|\theta|_\alpha$, $\alpha = 1,\dots,N$, of (5.20)

$$|x|^2 + 4|z| = |x|^2 + a|x|^2\mu(a|\theta|_\alpha) = (1 + a\mu(a|\theta|_\alpha))\frac{\sin^2(a|\theta|_\alpha)}{a^2|\theta|_\alpha^2}l_\alpha^2$$

by (5.20) and (5.7). Then

$$l_\alpha^2 = \frac{a^2|\theta|_\alpha^2}{\sin(a|\theta|_\alpha)\big(\sin(a|\theta|_\alpha) - \cos(a|\theta|_\alpha)\big) + a^2|\theta|_\alpha}(|x|^2 + 4|z|).$$

In the last simplest case it is easy to observe that if z is fixed, and $|x|$ tends to zero, then the ratio $\frac{4|z|}{a|x|^2}$ increases and the number of solutions of the equation $\frac{4|z|}{a|x|^2} = \mu(a|\theta|)$ also increases (see Figure 2). In this case, the function $\mu(a|\theta|_\alpha) = \frac{a|\theta|_\alpha - \cos(a|\theta|_\alpha)\sin(a|\theta|_\alpha)}{\sin^2(a|\theta|_\alpha)}$ tends to infinity as $|x| \to 0$, and we obtain that $\sin^2(a|\theta|) = 0$ and $a|\theta| = \pi n$, $n \in \mathbb{N}$. One sees that Theorem 5.2 is the limiting case of Theorem 5.4 as the ratio $\frac{4|z|}{a|x|^2}$ tends to ∞. If we fix x and let $|z|$ tend to 0, then the equation $\frac{4|z|}{a|x|^2} = \mu(a|\theta|)$ says that $\mu(a|\theta|) \to 0$. This implies that $|\theta| \to 0$ and we obtain Theorem 5.1 as another limit case of Theorem 5.4.

The last particular case is when $a_{m1} = a_{m2} = \ldots = a_{mn} = a_m$. We denote $|\theta|_l^2 = a_1^2\theta_1^2 + a_2^2\theta_2^2 + a_3^2\theta_3^2 = |\theta|_a^2$. The equation to find $|\theta|_a$ is

$$\mu(|\theta|_a) = \frac{4}{|x(1)|^2}\sqrt{\frac{z_m^2(1)}{a_m^2}}.$$

The value of θ_m and the lengths of geodesics are

$$\theta_m = \frac{z_m(1)|\theta|_a}{a_m^2\sqrt{\frac{z_m^2(1)}{a_m^2}}}, \qquad l^2(|\theta|_a) = |\theta|_a\Big(4\sqrt{\frac{z_m^2(1)}{a_m^2}} + |x(1)|^2\cot(|\theta|_a)\Big).$$

In the following theorem we consider the connection between the origin and a point $P(x,z)$ when some of the coordinates x_l vanish.

Theorem 5.6. *Given a point $P(x,z)$ with $x_l \neq 0$, $l = 1,\dots,p-1$, and $x_l = 0$, $l = p,\dots,n$, $z \neq 0$, there are infinitely many geodesics joining the point $O(0,0)$ with a point P. Let $S_{1m} = \sum_{l=1}^{p-1}\frac{a_{ml}^2|x_l(1)|^2}{|\theta|_l}\mu(|\theta|_l)$, $S_{2m} = \sum_{l=p}^{n}\frac{a_{ml}^2|\dot{x}_l(0)|^2}{\pi^2 n_l^2}$, $\mathbf{n}_\beta = (n_p,\dots,n_n)$ be a multiindex with positive integer-valued components for each $\beta \in \mathbb{N}$, and $\vartheta_\kappa = (|\theta|_1,\dots,|\theta|_{p-1})$, $\kappa = 1,\dots,N$ be solutions of the system*

$$|\theta|_l^2 = \sum_{m=1}^{3}\frac{16z_m^2(1)a_{ml}^2}{\big(S_{1m} + S_{2m}\big)^2}, \quad l = 1,\dots,p-1. \tag{5.21}$$

Then the equations of geodesics are

$$
\begin{aligned}
x_l^{(\kappa)}(s) =& \big(4\cot(|\theta|_l)\sin^2(s|\theta|_l) - 2\sin(2s|\theta|_l)\big)\frac{[\mathbf{Z}]_l}{|\theta|_l}x_l(1) \\
&+ \big(\frac{1}{2}\cot|\theta|_l\sin(2s|\theta|_l) + \sin^2(s|\theta|_l)\big)\mathcal{U}x_l(1), \quad l = 1,\dots,n, \quad \kappa = 1,2,\dots,N,
\end{aligned}
\tag{5.22}
$$

$$
x_l^{(\mathbf{n}_\beta)}(s) = 2\frac{1-\cos(2s\pi n_l)}{(\pi n_l)^2}[\mathbf{Z}]_l\dot{x}_l(0) + \frac{\sin(2s\pi n_l)}{2\pi n_l}\mathcal{U}\dot{x}_l(0), \quad l = 1,\dots,n, \quad \beta\in\mathbb{N},
\tag{5.23}
$$

where

$$
[\mathbf{Z}]_l = \begin{bmatrix}
0 & \frac{z_1a_{1l}}{S_{11}+S_{21}} & -\frac{z_3a_{3l}}{S_{13}+S_{23}} & -\frac{z_2a_{2l}}{S_{12}+S_{22}} \\
-\frac{z_1a_{1l}}{S_{11}+S_{21}} & 0 & -\frac{z_2a_{2l}}{S_{12}+S_{22}} & \frac{z_3a_{3l}}{S_{13}+S_{23}} \\
\frac{z_3a_{3l}}{S_{13}+S_{23}} & \frac{z_2a_{2l}}{S_{12}+S_{22}} & 0 & \frac{z_1a_{1l}}{S_{11}+S_{21}} \\
\frac{z_2a_{2l}}{S_{12}+S_{22}} & -\frac{z_3a_{3l}}{S_{13}+S_{23}} & -\frac{z_1a_{1l}}{S_{11}+S_{21}} & 0
\end{bmatrix},
\tag{5.24}
$$

and

(5.25)

$$
z_m^{(\kappa,\mathbf{n}_\beta)}(s) = \frac{z_m(1)}{S_{1m}+S_{2m}}\Big(\sum_{l=1}^{p-1}\frac{a_{ml}^2|x(1)|_l^2}{\sin^2(|\theta|_l)}\big(s - \frac{\sin(2s|\theta|_l)}{2|\theta|_l}\big)\big) + \sum_{l=p}^{n}\frac{a_{ml}^2|\dot{x}_l(0)|^2}{\pi^2n_l^2}\big(s - \frac{\sin(2s\pi n_l)}{2\pi n_l}\big)\big)\Big),
$$

with $m = 1,2,3$.

The lengths of these geodesics are

$$
l_{\kappa,\mathbf{n}_\beta}^2 = \sum_{l=1}^{n}|\dot{x}_l(0)|^2 = 16\sum_{m=1}^{3}\frac{z_m^2(1)}{S_{1m}+S_{2m}} + \sum_{l=1}^{p-1}|x_l(1)|^2|\theta|_l\cot(|\theta|_l).
\tag{5.26}
$$

Proof. If $x_l(1) = 0$, then the formula $|x_l(1)|^2 = \frac{\sin^2(|\theta|_l)}{|\theta|_l^2}|\dot{x}_l(0)|^2$ implies that $|\dot{x}_l(0)| = 0$ or $\sin^2(|\theta|_l) = 0$. If $|\dot{x}_l(0)| = 0$, then the corresponding $x_l(s)\equiv 0$. The more interesting case when $|\dot{x}_l(0)|\neq 0$ for $l = p,\dots,n$. Then $|\theta|_l = \pi n_l$, $n_l\in\mathbb{N}$, $l = p\dots,n$. We deduce from (4.22) for $s = 1$

$$
z_m(1) = \frac{\theta_m}{4}\Big(\sum_{l=1}^{p-1}\frac{a_{ml}^2|x_l(1)|^2}{|\theta|_l}\mu(|\theta|_l) + \sum_{l=p}^{n}\frac{a_{ml}^2|\dot{x}_l(0)|^2}{\pi^2n_l^2}\Big) = \frac{\theta_m}{4}(S_{1m}+S_{2m}),
\tag{5.27}
$$

where the number n_l can have any positive integer value. We conclude that the sum $S_{2m} = \sum_{l=p}^{n}\frac{a_{ml}^2|\dot{x}_l(0)|^2}{\pi^2n_l^2}$ admits countably many values. To define $|\theta|_l$ we find $\theta_m = \frac{4z_m(1)}{S_{1m}+S_{2m}}$, $m = 1,2,3$, from (5.27) and then argue as in Theorem 5.4:

$$
|\theta|_l^2 = \sum_{m=1}^{3}\theta_m^2a_{ml}^2 = \sum_{m=1}^{3}\frac{16z_m^2(1)a_{ml}^2}{\big(S_{1m}+S_{2m}\big)^2}, \quad l = 1,\dots,p-1.
\tag{5.28}
$$

Conclude, that for each multiindex with positive integer-valued components $\mathbf{n}_\beta = (n_p,\dots,n_n)$, $\beta\in\mathbb{N}$, the equation (5.28) defines the multiindex $\vartheta_\kappa = (|\theta|_1,\dots,|\theta|_{p-1})$, $\kappa = 1,\dots,N$. Let us fix one of the solutions $(\vartheta_\kappa,\mathbf{n}_\beta) = (|\theta|_1,\dots,|\theta|_{p-1},n_p,\dots,n_n)$. The relations (4.20) for $s = 1$ give

$$
\dot{x}_l(0) = \Big((|\theta|_l\cot|\theta|_l)\mathcal{U} - [\mathbf{M}]_l\Big)x_l(1), \quad l = 1,\dots,p-1.
\tag{5.29}
$$

Substituting (5.29) into (4.20) and (4.22), we obtain (5.22), (5.23), and (5.25). We get the form of (5.24) from $\theta_m = \frac{4z_m(1)}{S_{1m}+S_{2m}}$, $m = 1,2,3$ and the definition of the matrix $\mathbf{M}$.

To calculate the length of the geodesic we argue as follows:

$$
\begin{aligned}
\sum_{m=1}^{3} z_m\theta_m &= \frac{1}{4}\Big(\sum_{m=1}^{3}\theta_m^2 a_{ml}^2\sum_{l=1}^{p-1}\frac{|x_l(1)|^2\mu(|\theta|_l)}{|\theta|_l}+\sum_{m=1}^{3}\theta_m^2 a_{ml}^2\sum_{l=p}^{n}\frac{|\dot{x}_l(0)|^2}{\pi^2 n_l^2}\Big)\\
&= \frac{1}{4}\sum_{l=1}^{p-1}|x_l(1)|^2|\theta|_l\mu(|\theta|_l)+\sum_{l=p}^{n}\frac{|\dot{x}_l(0)|^2}{4}\\
&= \frac{1}{4}\sum_{l=1}^{p-1}\frac{|x_l(1)|^2|\theta|_l^2}{\sin^2(|\theta|_l)}-\frac{1}{4}\sum_{l=1}^{p-1}|x_l(1)|^2|\theta|_l\cot(|\theta|_l)+\sum_{l=p}^{n}\frac{|\dot{x}_l(0)|^2}{4}\\
&= \frac{|\dot{x}(0)|^2}{4}-\frac{1}{4}\sum_{l=1}^{p-1}|x_l(1)|^2|\theta|_l\cot(|\theta|_l).
\end{aligned}
\tag{5.30}
$$

From the other hand, since $\theta_m = \frac{4z_m(1)}{S_{1m}+S_{2m}}$, $m = 1, 2, 3$, we deduce

$$
\sum_{m=1}^{3} z_m\theta_m = 4\sum_{m=1}^{3}\frac{z_m^2(1)}{S_{1m}+S_{2m}}.
\tag{5.31}
$$

The formula (5.26) follows from (5.30) and (5.31). □

Remark 5.7. Let make some simulations for the anisotropic group Q^2. Set

$$
x_1(1) = (x_{11}, x_{12}, 0, 0), \quad x_2(1) = 0, \quad \dot{x}_2(0) = (\dot{x}_{21}(0), \dot{x}_{22}(0), 0, 0),
$$
$$
z_1(1) \neq 0, \qquad z_2(1) = z_3(1) = 0.
$$

In this case the equation (5.21) can be written in the form

$$
\mu(|\theta|_1) = \frac{4|z_1|}{a_{11}|x_1(1)|^2} - \frac{|\theta|_1 a_{12}^2|\dot{x}_2(0)|^2}{\pi^2 n^2 a_{11}^2|x_1(1)|^2}.
\tag{5.32}
$$

We present the solutions for different values of n: $n = 1, 2, 50$ in Figure 3. We see that for sufficiently big value of n the second term in the right hand side of (5.32) goes to 0, and we obtain a finite number of solutions for $|\theta|_1$.

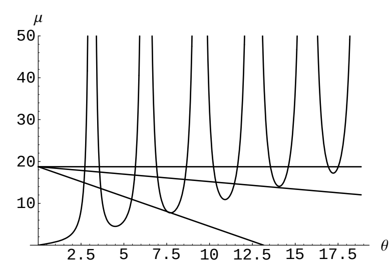

FIGURE 3. Solutions of equation (5.32)

Moreover, we obtain countably many geodesics, because of the second part of multiindex, corresponding to the positive integer values is countably infinite. Nevertheless, since the sums S_{2m} tends to 0 and the sums S_{1m} are strictly positive as $\mathbf{n} \to \infty$, we conclude that the lengths of these geodesics are bounded from the above. The projection in each subspace x_l are still ellipsoids. In Figures 4 and 5 we present the projection of a geodesic into spaces (x_1, z_1) and (x_2, z_1). We can see that the number of loops is different and increases in the subspace corresponding vanishing value of $x_l(1)$.

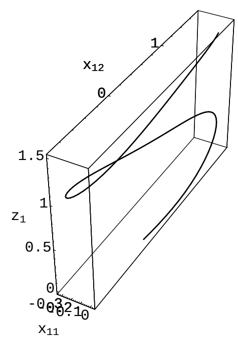

FIGURE 4. Projection of a geodesic to the space (x_1, z_1)

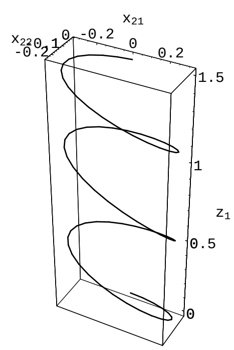

FIGURE 5. Projection of a geodesic to the space (x_2, z_1)

6. Complex Hamiltonian mechanics

Our aim now is to study the complex action which may be used to obtain the length of real geodesics.

Definition 6.1. A complex geodesic is the projection of a solution of the Hamiltonian system (4.3) with the non-standard boundary conditions

$$x(0) = 0, \quad x(1) = x, \quad z(0) = 0, \quad z(1) = z, \quad \text{and}$$

$$\theta_m = -i\tau_m, \quad m = 1, 2, 3,$$

on the (x, z)-space.

Let us introduce the notation $-i\tau$ for the vector $(-i\tau_1, -i\tau_2, -i\tau_3)$. We write $|\tau|_l = \sqrt{a_{1l}^2\tau_1^2 + a_{2l}^2\tau_2^2 + a_{3l}^3\tau_3^2}$. Then $|\theta|_l = \sqrt{a_{1l}^2\theta_1^2 + a_{2l}^2\theta_2^2 + a_{3l}^2\theta_3^2} = i|\tau|_l$.

Notice, that we should treat the missing directions apart from the directions in the underlying space.

Definition 6.2. The modifying complex action is defined as

$$f(x,z,\tau)=-i\sum_m \tau_m z_m+\int_0^1\big((\dot{x},\xi)-H(x,z,\xi,\tau)\big)\,ds. \tag{6.1}$$

We present some useful calculations following from the system (4.3).

$$\begin{aligned}(\xi,\dot{x})&=2|\xi|^2+(\mathbf{M}x,\xi)=\frac{1}{2}|\dot{x}|^2-\frac{1}{2}(\mathbf{M}x,\dot{x}),\\ |\xi|^2&=\frac{|\dot{x}|^2}{4}-\frac{1}{2}(\mathbf{M}x,\dot{x})+\frac{1}{4}(\mathbf{M}x,\mathbf{M}x)=\frac{|\dot{x}|^2}{4}-\frac{1}{2}(\mathbf{M}x,\dot{x})+\frac{1}{4}(\mathbf{\Theta}^2x,x),\\ (\mathbf{M}x,\xi)&=\frac{1}{2}(\mathbf{M}x,\dot{x})-\frac{1}{2}(\mathbf{\Theta}^2x,x).\end{aligned} \tag{6.2}$$

Making use of the formulas (4.2), (6.2), and (5.7), we deduce

$$\begin{aligned}f(x,z,\tau)=&-i\sum_m \tau_m z_m+\int_0^1\big((\dot{x},\xi)-H(x,z,\xi,\tau)\big)\,ds\\ =&-i\sum_m \tau_m z_m+\int_0^1\Big(\frac{|\dot{x}(s)|^2}{4}-\frac{1}{2}(\mathbf{M}x,\dot{x})\Big)\,ds\\ =&-i\sum_m \tau_m z_m+\sum_{l=1}^n\frac{|\dot{x}_l(0)|^2}{4}\int_0^1\cosh(2s|\tau|_l)\,ds\\ =&-i\sum_m \tau_m z_m+\sum_{l=1}^n\frac{|x_l|^2}{4}\frac{(i|\tau|_l)^2}{\sin^2(-i|\tau|_l)}\frac{\sinh(2|\tau|_l)}{2|\tau|_l}\\ =&-i\sum_m \tau_m z_m+\sum_{l=1}^n\frac{|x_l|^2}{4}|\tau|_l\coth|\tau|_l.\end{aligned}$$

The complex action function satisfies the Hamilton-Jacobi equation

$$\sum_{m=1}^3\tau_m\frac{\partial f}{\partial\tau_m}+H(x,z,\nabla_xf,\nabla_zf)=f. \tag{6.3}$$

Indeed, we have

$$\frac{\partial f}{\partial\tau_m}=-iz_m-i\tau_m\sum_{l=1}^n\frac{a_{ml}^2|x_l|^2}{4|\tau|_l}\mu(i|\tau|_l),\qquad m=1,2,3.$$

$$H(x,z,\frac{\partial f}{\partial x},\frac{\partial f}{\partial z})=H(x,z,\xi,\tau)=\sum_{l=1}^n\frac{|x_l|^2}{4}\frac{|\tau|_l^2}{\sinh^2|\tau|_l}$$

from (4.2), (6.2), and (4.19). Then,

$$\begin{aligned}\sum_{m=1}^3\tau_m\frac{\partial f}{\partial\tau_m}+H(x,z,\frac{\partial f}{\partial x},\frac{\partial f}{\partial z})&=-i\sum_m\tau_m z_m+\sum_{l=1}^n\frac{|x_l|^2|\tau|_l}{4}\Big(-i\mu(i|\tau|_l)+\frac{|\tau|_l}{\sinh^2|\tau|_l}\Big)\\ &=-i\sum_m\tau_m z_m+\sum_{l=1}^n\frac{|x_l|^2}{4}|\tau|_l\coth|\tau|_l=f.\end{aligned}$$

In the critical points τ_c, where $\frac{\partial f}{\partial \tau_m}=0$ we have from (5.17)

$$f(x,z,\tau_c)=H(x,z,\nabla_x f,\nabla_z f)=\frac{\mathcal{E}}{2}=\frac{l^2}{4}(\gamma),$$

where a geodesic curve γ connects the origin with (x,z).

7. Green's function for the Schrödinger operator

Consider the Schrödinger operator

$$L=\Delta_0-i\frac{\partial}{\partial u}.$$

We are looking for a distribution $P=P(x,z,u)$ on $\mathbb{R}^{4n}_x\times\mathbb{R}^3_z\times\mathbb{R}^+_u$ satisfying the following conditions

1) $LP=\Delta_0 P-i\frac{\partial P}{\partial u}=0$ for $u>0$,
2) $\lim\limits_{u\to 0^+}P(x,z,u)=\delta(x)\delta(z)$,

where δ stands for the Dirac distribution.

The next propositions are easily verified.

Proposition 7.1. *For any smooth function φ and any smooth vector fields $X_1,\dots,X_n$ we have*

$$\Delta e^{\varphi}=e^{\varphi}(\Delta\varphi+|\nabla\varphi|^2),$$

where $\Delta=\sum_{j=1}^n X_j^2$ and $|\nabla\varphi|^2=\sum_{j=1}^2(X_j\varphi)^2$.

We recall that X denotes the horizontal gradient $X_{11},\dots,X_{4\,n}$.

Proposition 7.2.

$$H(x,z,\nabla_x f,\nabla_z f)=|Xf|^2=\sum_{k=1}^4\sum_{l=1}^n\big(X_{kl}f\big)^2.$$

Proposition 7.3. *Let V and f be smooth functions of x, z, τ, and $X_1,\dots,X_n$ smooth vector fields. Then for any number p, we have the following identity:*

$$\Delta(Vf^{-p})=(\Delta V)f^{-p}-pf^{-p-1}\Big[(\Delta f)V+2(\nabla f)(\nabla V)\Big]+(-p)(-p-1)f^{-p-2}V|\nabla f|^2. \tag{7.1}$$

Proof. The formula (7.1) is obtained by the direct calculation. □

Before we go further, let us make some calculations. We apply Proposition 7.1 to $e^{-\frac{if}{u}}$ and system (2.3) of horizontal vector fields X_{kl}, $k=1,\dots,4$, $l=1,\dots,n$. Introducing the notation $\varphi=-\frac{if}{u}$, we get $\Delta_0\varphi=-\frac{i}{u}\Delta_0 f$, $|X\varphi|^2=-\frac{1}{u^2}|Xf|^2$, and

$$\Delta_0 e^{\varphi}=e^{\varphi}\Big(-\frac{i}{u}\Delta_0 f-\frac{1}{u^2}|Xf|^2\Big)=\frac{e^{\varphi}V(\tau)u^{2n+3}}{u^{2n+4}V(\tau)}\Big(-i\Delta_0 f-\frac{1}{u}|Xf|^2\Big). \tag{7.2}$$

Hamilton-Jacobi equation (6.3), Proposition 7.2, and the equality (7.2) imply

$$\Delta_0\frac{e^{\varphi}V(\tau)}{u^{2n+3}}=\frac{e^{\varphi}V(\tau)}{u^{2n+4}}\Big(-i\Delta_0 f-\frac{f}{u}+\frac{1}{u}\sum_{m=1}^3\tau_m\frac{\partial f}{\partial\tau_m}\Big). \tag{7.3}$$

Differentiating $\frac{e^{\varphi}V(\tau)}{u^{2n+3}}$ with respect to u, we obtain

$$-i\frac{\partial}{\partial u}\Big(\frac{e^{\varphi}V(\tau)}{u^{2n+3}}\Big)=\frac{e^{\varphi}V(\tau)}{u^{2n+4}}\Big(\frac{f}{u}+i(2n+3)\Big). \tag{7.4}$$

Summing (7.3) and (7.4), we have

$$\Big(\Delta_0 - i\frac{\partial}{\partial u}\Big)\frac{e^{\varphi}V(\tau)}{u^{2n+3}} = i\frac{e^{\varphi}V(\tau)}{u^{2n+4}}\Big((2n+3) - \Delta_0 f - \frac{i}{u}\sum_{m=1}^{3}\tau_m\frac{\partial f}{\partial \tau_m}\Big). \tag{7.5}$$

We express $-\frac{i}{u}\sum_{m=1}^{3}\tau_m\frac{\partial f}{\partial \tau_m}$ from the formula

$$\sum_{m=1}^{3}\frac{\partial}{\partial \tau_m}\big(e^{\varphi}V(\tau)\tau_m\big) = e^{\varphi}V(\tau)\Big(-\frac{i}{u}\sum_{m=1}^{3}\tau_m\frac{\partial f}{\partial \tau_m}\Big) + e^{\varphi}\sum_{m=1}^{3}\tau_m\frac{\partial V(\tau)}{\partial \tau_m} + 3e^{\varphi}V(\tau)$$

and put it into (7.5). Finally, we deduce

$$\begin{aligned}\Big(\Delta_0 - i\frac{\partial}{\partial u}\Big)\frac{e^{\varphi}V(\tau)}{u^{2n+3}} =& i\frac{e^{\varphi}}{u^{2n+4}}\Big((2n-\Delta f)V - \sum_{m=1}^{3}\tau_m\frac{\partial V}{\partial \tau_m}\Big)\\ &+\frac{i}{u^{2n+4}}\sum_{m=1}^{3}\frac{\partial}{\partial \tau_m}\big(e^{\varphi}V(\tau)\tau_m\big).\end{aligned} \tag{7.6}$$

The equation

$$(2n-\Delta f)V - \sum_{m=1}^{3}\tau_m\frac{\partial V}{\partial \tau_m} = 0 \tag{7.7}$$

is called the *transport equation*. We show that the function

$$V(\tau) = \prod_{l=1}^{n}\frac{|\tau|_l^2}{\sinh^2|\tau|_l}$$

is a solution of transport equation. Indeed, since

$$f = f(x,z,\tau) = -i\sum_m \tau_m z_m + \sum_{l=1}^{n}\frac{|x_l|^2}{4}|\tau|_l\coth(|\tau|_l),$$

we have

$$\frac{\partial f}{\partial z_m} = -i\tau_m, \quad \frac{\partial^2 f}{\partial z_m^2} = 0, \quad m=1,2,3.$$

$$\frac{\partial f}{\partial x_{kl}} = \frac{1}{2}x_{kl}|\tau|_l\coth(|\tau|_l), \quad \frac{\partial^2 f}{\partial x_{kl}^2} = \frac{|\tau|_l}{2}\coth(|\tau|_l), \quad k=1,\dots,4,\ l=1,\dots,n.$$

Finally,

$$\Delta f = 2\sum_{l=1}^{n}|\tau|_l\coth(|\tau|_l)$$

and

$$(2n-\Delta f)\,V(\tau) = 2V(\tau)\big(n - \sum_{l=1}^{n}|\tau|_l\coth(|\tau|_l)\big). \tag{7.8}$$

On the other hand the equalities

$$\begin{aligned}\frac{\partial V}{\partial \tau_m} &= \sum_{r=1}^{n}\prod_{l=1,\,l\neq r}^{n}\frac{|\tau|_l^2}{\sinh^2(|\tau|_l)}\cdot\frac{\partial}{\partial \tau_m}\Big(\frac{|\tau|_r^2}{\sinh^2(|\tau|_r)}\Big)\\ &= \sum_{r=1}^{n}\prod_{l=1,\,l\neq r}^{n}\frac{|\tau|_l^2}{\sinh^2(|\tau|_l)}\cdot\frac{2a_{mr}^2\tau_m}{\sinh^2|\tau|_r}\big(1-|\tau|_r\coth(|\tau|_r)\big), \quad m=1,2,3,\end{aligned}$$

imply

$$\begin{aligned}\sum_{m=1}^{3}\tau_m\frac{\partial V}{\partial\tau_m} &= \sum_{r=1}^{n}\prod_{l=1,\,l\neq r}^{n}\frac{|\tau|_l^2}{\sinh^2(|\tau|_l)}\cdot\frac{2|\tau|_r^2}{\sinh^2(|\tau|_r)}(1-|\tau|_r\coth(|\tau|_r))\\ &= 2\prod_{l=1}^{n}\frac{|\tau|_l^2}{\sinh^2(|\tau|_l)}\cdot\sum_{r=1}^{n}(1-|\tau|_r\coth(|\tau|_r)) = 2V(\tau)(n-\sum_{r=1}^{n}|\tau|_r\coth(|\tau|_r)),\end{aligned}$$

that shows that $V(\tau)$ is a solution of the transport equation (7.7). The function $V(\tau)$ is called the *volume element*.

If the volume element $V(\tau)$ satisfies the equation (7.7) then the equation (7.6) is reduced to the next one

$$\Big(\Delta_0 - i\frac{\partial}{\partial u}\Big)\frac{e^{\varphi}V(\tau)}{u^{2n+3}} = \frac{i}{u^{2n+4}}\sum_{m=1}^{3}\frac{\partial}{\partial\tau_m}\big(e^{\varphi}V(\tau)\tau_m\big). \tag{7.9}$$

We note that the expression $e^{\varphi}V(\tau)\tau_m$ vanishes as $|\tau|\to\infty$. Integrating over $\mathbb{R}^3$ with respect to $d\tau = d\tau_1 d\tau_2 d\tau_3$, we obtain

$$\Big(\Delta_0 - i\frac{\partial}{\partial u}\Big)\int_{\mathbb{R}^3}\frac{e^{\varphi}V(\tau)}{u^{2n+3}}\,d\tau = 0 \quad \text{for} \quad u>0.$$

Thus, the function

$$P(x,z,u) = \frac{C}{u^{2n+3}}\int_{\mathbb{R}^3} e^{\frac{-if}{u}}V(\tau)\,d\tau$$

satisfies the first condition to the Green function at the origin of the Schrödinger operator.

7.4. **The heat kernel on Q^n.** In this section we denote the time variable by t and we will consider the heat operator

$$\Delta_0 - \frac{\partial}{\partial t} = \sum_{k,l} Y_{kl}^2 - \frac{\partial}{\partial t},$$

where $Y = (Y_{1\,1},\dots,Y_{4\,n}) = \nabla_y + \frac{1}{2}(\sum_{m=1}^{3}\mathbf{M}_m y\frac{\partial}{\partial w_m})$ with $y=(y_{1\,1},\dots,y_{4\,n})$. The fundamental solution at the origin is the function $P(y,w,t)$ defined on $Q^n\times\mathbb{R}^1_+$ such that the following conditions

1) $\Delta_0 P - i\frac{\partial P}{\partial t} = 0 \quad \text{for} \quad t>0,$
2) $\lim\limits_{t\to 0^+} P(y,w,t) = \delta(y)\delta(w)$

hold. With the change of variables

$$u = it,\quad x_{kl} = iy_{kl},\quad w_m = z_m,\quad m=1,2,3,\quad k=1,\dots,4,\quad l=1,\dots,n,$$

the heat operator transforms to the Schrödinger operator

$$\Delta_0 - i\frac{\partial}{\partial u} = \sum_{k,l} X_{kl}^2 - i\frac{\partial}{\partial u}.$$

Indeed, under this change of variables we obtain

$$-i\frac{\partial}{\partial u} = \frac{\partial}{\partial t} \qquad \text{and} \qquad X = -iY.$$

The calculus of the previous subsection give us the following statement.

Theorem 7.1. *The heat kernel at the origin is given by*

$$P(y,w,t)=\frac{C}{t^{2n+3}}\int_{\mathbb{R}^3}e^{\frac{-f}{t}}V(\tau)\,d\tau,$$

where

$$f(y,w,\tau)=-i\sum_m\tau_m w_m+\sum_{l=1}^n\frac{|y_l|^2}{4}|\tau|_l\coth(|\tau|_l)$$

is the modified complex action and

$$V(\tau)=\prod_{l=1}^n\frac{|\tau|_l^2}{\sinh^2(|\tau|_l)}$$

is the volume element.

7.5. **Green function for the sub-Laplace operator.** Let us integrate the kernel $P(x,z,u)$ with respect to the time variable u on $(0,\infty)$. That is

$$\begin{aligned}\int_0^\infty P(x,z,u)\,du=&\int_0^\infty\frac{C}{u^{2n+3}}\int_{\mathbb{R}^3}e^{-if/u}V(\tau)\,d\tau\ du\\ =&C\int_{\mathbb{R}^3}V(\tau)\Big(\int_0^\infty u^{-2n-3}e^{-if/u}\,du\Big)\,d\tau.\end{aligned}$$

We first look at the inner integral:

$$\int_0^\infty u^{-2n-3}e^{-if/u}\,du.$$

Changing variable $v=\frac{if}{u}$, $u=\frac{if}{v}$, yields $dv=\frac{-if}{u^2}\,du$ and $du=-\frac{if}{v^2}\,dv$. Hence, we have

$$\int_0^\infty u^{-2n-3}e^{-if/u}\,du=\frac{1}{i^{2n+2}f^{2n+2}}\int_0^\infty e^{-v}v^{2n+3-2}\,dv=\frac{\Gamma(2n+2)}{i^{2n+2}f^{2n+2}}.$$

Let us introduce the following notation

$$-G(x,z)=\int_0^\infty P(x,z,u)\,du=C\frac{\Gamma(2n+2)}{i^{2n+2}}\int_{\mathbb{R}^3}\frac{V(\tau)}{f^{2n+2}(x,z,\tau)}\,d\tau.$$

The aim of this section is to show that the function $-G(x,z)$ is the Green function for the sub-Laplacian operator. Firstly, we need some auxiliary results.

Proposition 7.6. *Denote*

$$f(x,z,w)=\sum_{l=1}^n\frac{1}{4}|x_l|^2|w|_l\coth(|w|_l)-i\sum_{m=1}^3 w_m z_m=\gamma(x,w)-i\sum_{m=1}^3 w_m z_m,$$

where $w=\tau+i\varepsilon\tilde{z}$. Then there exist positive constants c_1, c_2, and ε_0 such that for all real τ_m, all $0<\varepsilon<\varepsilon_0$, and all $x\in\mathbb{R}^{4n}$, $z=(z_1,z_2,z_3)\in\mathbb{R}^3$ we have the estimates

$$\big|Im(\gamma)(x,\tau+i\varepsilon\tilde{z})\big|\le c_1\varepsilon|x|^2, \tag{7.10}$$

$$Re(\gamma)(x,\tau+i\varepsilon\tilde{z})\ge c_2|x|^2, \tag{7.11}$$

$$Re(f)(x,z,\tau+i\varepsilon\tilde{z})\ge c_2(|x|^2+\varepsilon|z|). \tag{7.12}$$

Here $\tilde{z}=\frac{z}{|z|}$ if $z\neq 0$ and $\tilde{z}=0$ if $z=0$.

Proof. If $\tilde z = 0$, then $\mathrm{Im}(\gamma)(x,\tau) = 0$ and since $|\tau|_l \coth(|\tau|_l) \geq 1$, $l = 1,\dots,n$, we have $\mathrm{Re}(\gamma)(x,\tau) \geq \frac{|x|^2}{4}$.

Suppose that $\tilde z \neq 0$. We denote $|w|_l = \Big(\sum_{m=1}^3 a_{ml}^2\big(\tau_m + i\varepsilon\tilde z_m\big)^2\Big)^{1/2} = \alpha_l + i\beta_l$, where

$$\alpha_l = \Big(\big(|\tau|_l^2 - \varepsilon^2|\tilde z|_l^2\big)^2 + \big(2\varepsilon\sum_m a_{ml}^2\tau_m\tilde z_m\big)^2\Big)^{1/4}\cos\frac{\arctan\Big(\frac{2\varepsilon\sum_m a_{ml}^2\tau_m\tilde z_m}{|\tau|_l^2-\varepsilon^2|\tilde z|_l^2}\Big)}{2} + \pi d, \quad d = 0,1,$$

and

$$\beta_l = \Big(\big(|\tau|_l^2 - \varepsilon^2|\tilde z|_l^2\big)^2 + \big(2\varepsilon\sum_m a_{ml}^2\tau_m\tilde z_m\big)^2\Big)^{1/4}\sin\frac{\arctan\Big(\frac{2\varepsilon\sum_m a_{ml}^2\tau_m\tilde z_m}{|\tau|_l^2-\varepsilon^2|\tilde z|_l^2}\Big)}{2} + \pi d, \quad d = 0,1.$$

We consider the case $d = 0$, another one can be treated similarly. Since

$$\coth(\alpha + i\beta) = \frac{\sinh 2\alpha}{\cosh 2\alpha - \cos 2\beta} - i\frac{\sin 2\beta}{\cosh 2\alpha - \cos 2\beta},$$

we have

$$\begin{aligned}\mathrm{Re}(\gamma)(x,\tau + i\varepsilon\tilde z) + i\mathrm{Im}(\gamma)(x,\tau + i\varepsilon\tilde z) &= \sum_{l=1}^n \frac{|x_l|^2}{4}(\alpha_l + i\beta_l)\coth(\alpha_l + i\beta_l)\\ &= \sum_{l=1}^n \frac{|x_l|^2}{4}\Big(\frac{\alpha_l\sinh\alpha_l\cosh\alpha_l + \beta_l\sin\beta_l\cos\beta_l}{\sinh^2\alpha_l + \sin^2\beta_l}\Big)\\ &\quad + i\sum_{l=1}^n \frac{|x_l|^2}{4}\Big(\frac{\beta_l\sinh\alpha_l\cosh\alpha_l - \alpha_l\sin\beta_l\cos\beta_l}{\sinh^2\alpha_l + \sin^2\beta_l}\Big).\end{aligned}$$

Denotes by ψ_l the angle between nonzero vectors $(a_{1l}\tau_1, a_{2l}\tau_2, a_{3l}\tau_3)$ and $(a_{1l}\tilde z_1, a_{2l}\tilde z_2, a_{3l}\tilde z_3)$. We consider two cases, when $\cos\psi_l = 0$ for all $l = 1,\dots,n$, and $\cos\psi_l = \vartheta_l \neq 0$ for some index l.

Case 1. If $\cos\psi_l = 0$ for all $l = 1,\dots,n$, then $\sum_{m=1}^3 a_{ml}^2\tau_m\tilde z_m = 0$. We have

$$\alpha_l = \big(|\tau|_l^2 - \varepsilon^2|\tilde z|_l^2\big)^{1/2} \quad \text{and} \quad \beta_l = 0.$$

It gives

$$\mathrm{Im}(\gamma)(x,\tau + i\varepsilon\tilde z) = \sum_{l=1}^n \frac{|x_l|^2}{4}\Big(\frac{\beta_l\sinh\alpha_l\cosh\alpha_l - \alpha_l\sin\beta_l\cos\beta_l}{\sinh^2\alpha_l + \sin^2\beta_l}\Big) = 0,$$

$$\mathrm{Re}(\gamma)(x,\tau + i\varepsilon\tilde z) = \sum_{l=1}^n \frac{|x_l|^2}{4}\alpha_l\coth(\alpha_l) \geq \frac{|x|^2}{4}, \qquad \forall\alpha_l \in \mathbb{R},$$

because $\alpha_l\coth(\alpha_l) \geq 1$.

Case 2. If $\cos\psi_l = \vartheta_l \neq 0$ for some $l = 1,\dots,n$, then $\sum_{m=1}^3 a_{ml}^2\tau_m\tilde z_m = \vartheta_l|\tau|_l|\tilde z|_l$. We can suppose that ε satisfies $0 < \varepsilon^2 < \min_{l=1,\dots,n}\big\{\frac{|\tau|_l^2}{2|\tilde z|_l^2}\big\}$. We obtain

$$\tfrac{2\varepsilon\vartheta_l|\tilde z|_l}{|\tau|_l} < \tfrac{2\varepsilon\vartheta_l|\tau|_l|\tilde z|_l}{|\tau|_l^2-\varepsilon^2|\tilde z|_l^2} < \tfrac{4\varepsilon\vartheta_l|\tilde z|_l}{|\tau|_l},$$

$$\begin{aligned}k_1|\tau|_l &< \Big(\tfrac{|\tau|_l^4}{4} + \big(2\varepsilon\vartheta_l|\tau|_l|\tilde z|_l\big)^2\Big)^{1/4} < \Big(\big(|\tau|_l^2 - \varepsilon^2|\tilde z|_l^2\big)^2 + \big(2\varepsilon\textstyle\sum_m a_{ml}^2\tau_m\tilde z_m\big)^2\Big)^{1/4}\\ &< \Big(|\tau|_l^4 + \big(2\varepsilon\vartheta_l|\tau|_l|\tilde z|_l\big)^2\Big)^{1/4} < k_2(\vartheta)|\tau|_l.\end{aligned} \tag{7.13}$$

Now, we put one more restriction to ε assuming that $\varepsilon < \min_{l=1,\dots,n}\left\{\frac{|\tau|_l}{4\vartheta_l|\tilde z|_l}\right\}$. Then

$$\frac{2\varepsilon\vartheta_l|\tilde z|_l}{\pi|\tau|_l} \le \frac{1}{2}\arctan\frac{2\varepsilon\vartheta_l|\tilde z|_l}{|\tau|_l} \le \frac{1}{2}\arctan\frac{2\varepsilon\vartheta_l|\tau|_l|\tilde z|_l}{|\tau|_l^2-\varepsilon_l^2|\tilde z|_l^2} \le \frac{1}{2}\arctan\frac{4\varepsilon\vartheta_l|\tilde z|_l}{|\tau|_l} \le \frac{2\varepsilon\vartheta_l|\tilde z|_l}{|\tau|_l},$$

and we get

$$\frac{\sqrt{2}}{2} < \cos\Big(\frac{2\varepsilon\vartheta_l|\tilde z|_l}{|\tau|_l}\Big) < \cos\frac{\arctan\Big(\frac{2\varepsilon\vartheta_l|\tau|_l|\tilde z|_l}{|\tau|_l^2-\varepsilon^2|\tilde z|_l^2}\Big)}{2} < \cos\Big(\frac{2\varepsilon\vartheta_l|\tilde z|_l}{\pi|\tau|_l}\Big) < 1, \tag{7.14}$$

$$\frac{4\varepsilon\vartheta_l|\tilde z|_l}{\pi^2|\tau|_l} < \sin\Big(\frac{2\varepsilon\vartheta_l|\tilde z|_l}{\pi|\tau|_l}\Big) < \sin\frac{\arctan\Big(\frac{2\varepsilon\vartheta_l|\tau|_l|\tilde z|_l}{|\tau|_l^2-\varepsilon^2|\tilde z|_l^2}\Big)}{2} < \sin\Big(\frac{2\varepsilon\vartheta_l|\tilde z|_l}{|\tau|_l}\Big) < \frac{2\varepsilon\vartheta_l|\tilde z|_l}{|\tau|_l}.$$

We observe that $|\tilde z|_l^2 = \sum_{m=1}^3 a_{ml}^2\frac{z_m^2}{|z|^2} \le \sum_{m=1}^3 a_{ml}^2 \le \overline{a}$, where $\overline{a} = \max\limits_{m,l}\{a_{ml}^2\}$. From the other hand, if we denote $\underline{a} = \min\limits_{m,l}\{a_{ml}^2\}$, then $\underline{a} \le \sum_{m=1}^3 a_{ml}^2\frac{z_m^2}{|z|^2} = |\tilde z|_l^2$. From (7.13) and (7.14) we estimate the value of α_l and β_l as follows

$$k_1|\tau|_l < \alpha_l < k_2|\tau|_l, \tag{7.15}$$

$$k_3(\underline{a})\varepsilon < \beta_l < k_4(\overline{a})\varepsilon. \tag{7.16}$$

If $|\tau|_l < 1$ we use the Taylor decomposition and obtain

$$\Big|\frac{\beta_l\sinh\alpha_l\cosh\alpha_l - \alpha_l\sin\beta_l\cos\beta_l}{\sinh^2\alpha_l+\sin^2\beta_l}\Big| = \Big|\frac{-\frac{2}{3}\alpha_l\beta_l(\alpha_l^2-\beta_l^2)+O(\alpha_l^4-\beta_l^4)}{\alpha_l^2+\beta_l^2-O(\alpha_l^4+\beta_l^4)}\Big| \le k_5\varepsilon.$$

If $|\tau|_l \ge 1$ we argue as follows

$$\Big|\frac{\beta_l\sinh\alpha_l\cosh\alpha_l - \alpha_l\sin\beta_l\cos\beta_l}{\sinh^2\alpha_l+\sin^2\beta_l}\Big| \le |\beta_l|\big(|\coth(\alpha_l)| + \big|\frac{\alpha_l}{\sinh^2(\alpha_l)}\big|\big) \le k_6\varepsilon,$$

because α_l is bounded from below, the functions $|\coth(\alpha_l)|$ and $\big|\frac{\alpha_l}{\sinh^2\alpha_l}\big|$ are bounded from above. The last two estimates imply

$$|\mathrm{Im}(\gamma)(x,\tau+i\varepsilon\tilde z)| \le \sum_{n}^{l=1}\frac{|x|_l^2}{4}k_7\varepsilon \le c_1\varepsilon|x|^2.$$

To obtain (7.11) we change the arguments. Let us focus on the value of the derivatives $\frac{\partial\gamma(x,w)}{\partial w_m}$ at $w_m = i\zeta_m$, $\zeta_m \in \mathbb{R}$ for $m = 1,2,3$. The equality

$$\frac{\partial\gamma(x,w)}{\partial w_m}\Big|_{w=i\zeta} = -\sum_{l=1}^n\frac{|x|_l^2}{4}\frac{a_{ml}^2 w_m}{|w|_l}\Big(\frac{|w|_l}{\sinh^2(|w|_l)} - \coth(|w|_l)\Big)\Big|_{w=i\zeta} = i\sum_{l=1}^n\frac{|x|_l^2}{4}\frac{a_{ml}^2\zeta_m}{|\zeta|_l}\mu(|\zeta|_l)$$

implies $\frac{\partial\mathrm{Re}(\gamma(x,w))}{\partial w_m}\Big|_{w=i\zeta} = 0$ and we conclude that $w = i\zeta$ is a critical point for $\mathrm{Re}(\gamma(x,w))$. Let us look at the Hessian at $w = i\zeta$. We have

$$\frac{\partial^2\gamma}{\partial w_m^2}\Big|_{w=i\zeta} = \sum_{l=1}^n\frac{|x|_l^2 a_{ml}^2}{4}\Big[\frac{\mu(|\zeta|_l)}{|\zeta|_l}\big(1-\frac{a_{ml}^2\zeta_m^2}{|\zeta|_l^2}\big) + \frac{2a_{ml}^2\zeta_m^2}{|\zeta|_l^2\sin^2(|\zeta|_l)}\big(1-|\zeta|_l\cot(|\zeta|_l)\big)\Big].$$

Since $1-\frac{a_{ml}^2\zeta_m^2}{|\zeta|_l^2}\geq 0$ and $1-|\zeta|_l\cot(|\zeta|_l)\geq 0$ we see that $\frac{\partial^2\gamma}{\partial w_m^2}\Big|_{w=i\zeta}>0$ for $0\neq|\zeta|_l<\frac{\pi}{2}$, $m=1,2,3$. The mixed second derivatives are

$$\begin{aligned}\frac{\partial^2\gamma}{\partial w_m\partial w_k}\Big|_{w=i\zeta} &= \sum_{l=1}^n\frac{|x_l|^2}{4}\frac{a_{ml}^2a_{kl}^2\zeta_m\zeta_k}{|\zeta|_l^2}\Big[\frac{\cot(|\zeta|_l)}{|\zeta|_l}-\frac{2|\zeta|_l\cot(|\zeta|_l)}{\sin^2(|\zeta|_l)}+\frac{1}{\sin^2(|\zeta|_l)}\Big]\\ &= \sum_{l=1}^n\frac{|x_l|^2}{4}\frac{a_{ml}^2a_{kl}^2\zeta_m\zeta_k}{|\zeta|_l^2}g(|\zeta|_l),\end{aligned}$$

where $g(|\zeta|_l)=\frac{\cot(|\zeta|_l)}{|\zeta|_l}-\frac{2|\zeta|_l\cot(|\zeta|_l)}{\sin^2(|\zeta|_l)}+\frac{1}{\sin^2(|\zeta|_l)}$. We observe that since all second derivatives of $\gamma(x,w)$ are real at the critical point, the Hessian for $\gamma(x,w)$ coincides with the Hessian H for $\mathrm{Re}(\gamma(x,w))$ at $w=i\zeta$. We write $\frac{\partial^2\gamma}{\partial w_m^2}$ as

$$\frac{\partial^2\gamma}{\partial w_m^2}\Big|_{w=i\zeta}=\sum_{l=1}^n\frac{|x|_l^2}{4}\Big[\frac{a_{ml}^2\mu(|\zeta|_l)}{|\zeta|_l}+\frac{a_{ml}^4\zeta_m^2}{|\zeta|_l^2}g(|\zeta|_l)\Big].$$

Then the Hessian can be written in the form $H=\sum_{l=1}^n H_l$, where

$$H_l=\frac{|x|_l^2}{4}\begin{bmatrix}\frac{a_{1l}^2\mu(|\zeta|_l)}{|\zeta|_l}+\frac{a_{1l}^4\zeta_1^2}{|\zeta|_l^2}g(|\zeta|_l) & \frac{a_{1l}^2a_{2l}^2\zeta_1\zeta_2}{|\zeta|_l^2}g(|\zeta|_l) & \frac{a_{1l}^2a_{3l}^2\zeta_1\zeta_3}{|\zeta|_l^2}g(|\zeta|_l)\\ \frac{a_{1l}^2a_{2l}^2\zeta_1\zeta_2}{|\zeta|_l^2}g(|\zeta|_l) & \frac{a_{2l}^2\mu(|\zeta|_l)}{|\zeta|_l}+\frac{a_{2l}^4\zeta_2^2}{|\zeta|_l^2}g(|\zeta|_l) & \frac{a_{2l}^2a_{3l}^2\zeta_2\zeta_3}{|\zeta|_l^2}g(|\zeta|_l)\\ \frac{a_{1l}^2a_{3l}^2\zeta_1\zeta_3}{|\zeta|_l^2}g(|\zeta|_l) & \frac{a_{2l}^2a_{3l}^2\zeta_2\zeta_3}{|\zeta|_l^2}g(|\zeta|_l) & \frac{a_{3l}^2\mu(|\zeta|_l)}{|\zeta|_l}+\frac{a_{3l}^4\zeta_3^2}{|\zeta|_l^2}g(|\zeta|_l)\end{bmatrix}.$$

To show that H is positive definite we need to show that each H_l is positive definite. It was shown that

$$\frac{a_{1l}^2\mu(|\zeta|_l)}{|\zeta|_l}+\frac{a_{1l}^4\zeta_1^2}{|\zeta|_l^2}g(|\zeta|_l)>0,\quad\text{for}\quad 0\neq|\zeta|_l<\frac{\pi}{2}.$$

Then we have

$$\begin{aligned}&\Big(\frac{a_{1l}^2\mu(|\zeta|_l)}{|\zeta|_l}+\frac{a_{1l}^4\zeta_1^2}{|\zeta|_l^2}g(|\zeta|_l)\Big)\Big(\frac{a_{2l}^2\mu(|\zeta|_l)}{|\zeta|_l}+\frac{a_{2l}^4\zeta_2^2}{|\zeta|_l^2}g(|\zeta|_l)\Big)-\Big(\frac{a_{1l}^2a_{2l}^2\zeta_1\zeta_2}{|\zeta|_l^2}g(|\zeta|_l)\Big)^2\\ &=\frac{a_{1l}^2a_{2l}^2\mu^2(|\zeta|_l)}{|\zeta|_l^2}\Big(1-\frac{a_{1l}^2\zeta_1^2+a_{2l}^2\zeta_2^2}{|\zeta|_l^2}\Big)+\frac{2a_{1l}^2a_{2l}^2\mu(|\zeta|_l)(a_{1l}^2\zeta_1^2+a_{2l}^2\zeta_2^2)}{|\zeta|_l^4\sin^2(|\zeta|_l)}\big(1-|\zeta|_l\cot(|\zeta|_l)\big)>0\end{aligned}$$

for $0\neq|\zeta|_l<\frac{\pi}{2}$. Finally, we calculate $\det H_l$:

$$\frac{|x|_l^6}{4^3}\frac{a_{1l}^2a_{2l}^2a_{3l}^2\mu^2(|\zeta|_l)}{|\zeta|_l^2}\Big(\frac{\mu(|\zeta|_l)}{|\zeta|_l}+g(|\zeta|_l)\Big)=\frac{a_{1l}^2a_{2l}^2a_{3l}^2|x|_l^6}{4^3}\frac{2\mu^2(|\zeta|_l)}{|\zeta|_l^2\sin^2(|\zeta|_l)}\big(1-|\zeta|_l\cot(|\zeta|_l)\big)>0$$

for $0\neq|\zeta|_l<\frac{\pi}{2}$. We conclude that the Hessian is positive definite and $\mathrm{Re}(\gamma(z,w))$ has a local minimum at $w=i\zeta$. Thus

$$\mathrm{Re}(\gamma(x,w))\geq\mathrm{Re}(\gamma(x,w))|_{w=i\zeta}=\sum_{l=1}^n\frac{|x|_l^2}{4}|\zeta|_l\cot(|\zeta|_l)\geq c_2|x|^2\quad\text{if}\quad|\zeta|_l<\pi/4.$$

Put $\zeta=\varepsilon\tilde{z}$, then $|\zeta|_l\leq\varepsilon\overline{a}$ and (7.11) holds with $\varepsilon_0=\frac{\pi}{4\overline{a}}$.

Estimate (7.12) is a consequence of estimates (7.10) and (7.11) since

$$\begin{aligned}f(x,z,\tau+i\varepsilon\tilde{z}) &= \gamma(x,z,\tau+i\varepsilon\tilde{z})-i\sum_{m=1}^3(\tau_m+i\varepsilon\tfrac{z_m}{|z|})z_m\\ &= \gamma(x,z,\tau+i\varepsilon\tilde{z})+\varepsilon|z|-i\sum_{m=1}^3\tau_m z_m.\end{aligned}$$

$\square$

Theorem 7.6 in a non-diagonal situation was proved in [4].

Lemma 7.7. *If x is a non-zero vector in $\mathbb{R}^{4n}$, the integral*

$$\tilde{G}(x,z) = \int_{\mathbb{R}^3} \frac{V(\tau)}{f^{2n+2}(x,z,\tau)} d\tau \tag{7.17}$$

is absolutely convergent and one has for $x \neq 0$

$$\Delta_0 \tilde{G}(x,z) = 0.$$

Proof. Since the function $V(\tau)$ does not depend on x and z, we have $\Delta_0 V = 0$, $X_{kl} V = 0$, $k = 1,2,3,4$, $l = 1,\dots,n$, and the equation (7.1) reduced to the following one

$$\Delta_0(Vf^{-p}) = -p\Big(f^{-p-1} V \Delta_0 f + (-p-1) V f^{-p-2} H(Xf)\Big). \tag{7.18}$$

Here $H(Xf) = |Xf|^2$ by Proposition 7.2. Moreover, taking into account that the complex action function $f(x,z,\tau)$ satisfies the Hamilton-Jacobi equation (6.3) and $p = 2n+2$, we get (7.19)

$$\Delta_0(Vf^{-2n-2}) = (-2n-2)\Big(f^{-2n-3} V \Delta_0 f + (-2n-3) V f^{-2n-3} + (2n+3) V f^{-2n-4} \sum_{m=1}^{3} \tau_m \frac{\partial f}{\partial \tau_m}\Big).$$

Substituting the last term in the right hand side of (7.19) from the formula

$$\sum_{m=1}^{3} \frac{\partial}{\partial \tau_m}\big(\tau_m V f^{-2n-3}\big) = 3Vf^{-2n-3} + f^{-2n-3} \sum_{m=1}^{3} \tau_m \frac{\partial V}{\partial \tau_m} - (2n+3) V f^{-2n-4} \sum_{m=1}^{3} \tau_m \frac{\partial f}{\partial \tau_m},$$

we deduce

$$\Delta_0(Vf^{-2n-2}) = (-2n-2)\Big(f^{-2n-3}\big(V(\Delta_0 f - 2n) + \sum_{m=1}^{3} \tau_m \frac{\partial V}{\partial \tau_m}\big) - \sum_{m=1}^{3} \frac{\partial}{\partial \tau_m}\big(\tau_m V f^{-2n-3}\big)\Big).$$

Since the volume element $V(\tau)$ is a solution of the transport equation (7.7), finally, we obtain

$$\Delta_0 \tilde{G}(x,z) = \int_{\mathbb{R}^3} \Delta_0(Vf^{-2n-2})\, d\tau = (2n+2) \int_{\mathbb{R}^3} \sum_{m=1}^{3} \frac{\partial}{\partial \tau_m}\big(\tau_m V f^{-2n-3}\big)\, d\tau.$$

We observe that

$$V(\tau) = \prod_{l=1}^{n} \frac{|\tau|_l^2}{\sinh^2 |\tau|_l} \to 0 \quad \text{as one of the} \quad |\tau_m| = R \to \infty, \tag{7.20}$$

and

$$|f| \geq \sum_{l=1}^{n} \frac{|x_l|^2}{2} |\tau|_l \coth(|\tau|_l) \geq \frac{|x|^2}{2} \tag{7.21}$$

because of $|\tau|_l \coth(|\tau|_l) \geq 1$ for $l = 1,\dots,n$. The estimates (7.20) and (7.21) show

$$\lim_{R\to\infty} \int_{|\tau_m| \leq R} \frac{\partial}{\partial \tau_m}\big(\tau_m V f^{-2n-3}\big) = 0. \tag{7.22}$$

The last equality implies

$$\Delta_0(Vf^{-2n-2}) = (2n+2) \int_{\mathbb{R}^3} \sum_{m=1}^{3} \frac{\partial}{\partial \tau_m}\big(\tau_m V f^{-2n-3}\big)\, d\tau = 0,$$

that terminates the proof of Lemma 7.7. □

The argument of Lemma 7.7 is not valid for $x = 0$. In fact, the integral (7.17) is divergent because the denominator of the integrand contains $-\sum_{m=1}^{3} \tau_m z_m$ which is zero along a hyperplane of $\mathbb{R}^3$. We will treat the case by changing contour by adding a small imaginary part to the τ_m's. We shall prove that for $x \neq 0$ we can change the contour in (7.17) and when x will be zero the integral (7.17) will still be convergent on the new contour. In order to achieve this goal, we need to use Proposition 7.6.

Proposition 7.8. *For $x \neq 0$, the integral $\tilde{G}(x,z)$ defined in (7.17) is given by*

$$\tilde{G}(x,z) = \int_{\mathbb{R}^3} \frac{V(\tau + i\varepsilon\tilde{z})}{f^{2n+2}(x,z,\tau + i\varepsilon\tilde{z})} d\tau \tag{7.23}$$

for $0 < \varepsilon < \varepsilon_0$ sufficiently small. The integral (7.23) makes sense even for $x = 0$ and $z \neq 0$, so the function $\tilde{G}(x,z)$ is well-defined (in fact, real analytic) except at the origin in $\mathbb{R}^{4n} \times \mathbb{R}^3$ and satisfies

$$\Delta_0 \tilde{G}(x,z) = 0 \qquad \text{for} \quad (x,z) \neq (0,0).$$

Proof. We may prove this theorem by imitating the idea in [4]. Set

$$\Omega_{K,\varepsilon} = \{\xi = \tau + i\eta\tilde{z} : \tau \in \mathbb{R}^3,\ |\tau| < K,\ 0 < \eta < \varepsilon\} \subset \mathbb{C}^3.$$

Assume that $|x| \neq 0$. By Theorem 7.1 the differential form

$$\omega = \big(V(\zeta)/f^{2n+2}(x,z,\zeta)\big) d\xi_1 \wedge d\xi_2 \wedge d\xi_3$$

is a homomorphic form of type $(3,0)$ in $\Omega_{K,\varepsilon}$. It is easy to see that its differential is zero. Hence, by Stokes's Theorem

$$\int_{\partial\Omega_{K,\varepsilon}} \omega = 0.$$

The boundary can be written as $\partial\Omega_{K,\varepsilon} = \partial\Omega_1 \cup \partial\Omega_2 \cup \partial\Omega_3$.

(1) The set $\partial\Omega_1 = \{\tau \in \mathbb{R}^3 : |\tau| < K,\ \eta = 0\}$ which is such that

$$\lim_{K\to\infty} \int_{|\tau|<K} \frac{V(\tau)}{f^{2n+2}(x,z,\tau)} d\tau = \tilde{G}(x,z)$$

since the integral (7.17) converges absolutely.

(2) The set $\partial\Omega_2 = \{\tau + i\varepsilon\tilde{z} : |\tau| < K,\ \eta = \varepsilon\}$. The integral (7.23) converges absolutely by Proposition 7.6 in this case.

(3) The set $\partial\Omega_3 = \{\xi = \tau + i\varepsilon\tilde{z} : |\tau| = K,\ 0 < \eta < \varepsilon\}$. Again, by Proposition 7.6, one has

$$\lim_{K\to\infty} \int_{\partial\Omega_3} \frac{V(\tau + i\eta\tilde{z})}{f^{2n+2}(x,z,\tau + i\eta\tilde{z})} d\xi = 0.$$

By the discussion above, one may conclude that for $x \neq 0$, $\tilde{G}(x,z)$ is given by the integral (7.23) on a shifted contour. Moreover, this integral is absolutely convergent even when $x = 0$ and $z \neq 0$ by Proposition 7.6. We complete the proof of this theorem. □

Theorem 7.2. *The kernel $G(x,z)$ of the Green's function for the sub-Laplacian Δ_0 is given by the formula*

$$G(x,z) = -\frac{2^{2n}(2\pi)^{2n+3}}{(2n+1)!} \int_{\mathbb{R}^3} \frac{V(\tau + i\varepsilon\tilde{z})}{f^{2n+2}(\tau + i\varepsilon\tilde{z})} d\tau.$$

Proof. For any $K > 0$, denote

$$\tilde{G}_K(x,z) = \frac{1}{\Gamma(2n+2)} \int_{\mathbb{R}^3} V(\tau + i\varepsilon\tilde{z})d\tau \int_0^K t^{2n+2-1} e^{-tf} dt.$$

The function $\tilde{G}_K(x,z)$ is smooth everywhere and for $(x,z) \neq (0,0)$, one has

$$\lim_{K\to\infty} \tilde{G}_K(x,z) = \tilde{G}(x,z).$$

Using Proposition 7.6, we know that

$$\left|\tilde{G}_K(x,z)\right| \leq C \int_{\mathbb{R}^3} d\tau \int_0^\infty \prod_{l=1}^n \frac{|\tau|_l^2}{\sinh^2 |\tau|_l} \cdot e^{-c_1(|x|^2+|z|)t} t^{2n+1} dt \leq C(|x|^2 + |z|)^{-2n-2}.$$

But the function $(|x|^2+|z|)^{-2n-2} \in L^1_{loc}(\mathbb{R}^{4n+3})$ since the homogeneous degree is $4n+6$ in this case. It follows that

$$\lim_{K\to\infty} \tilde{G}_K(x,z) = \tilde{G}(x,z) \qquad \text{in} \quad L^1_{loc}(\mathbb{R}^{4n+3})$$

by the Dominated Convergence Theorem. We first calculate $\Delta_0 \tilde{G}_R(x,z)$ for $x \neq 0$. We need to compute $\Delta_0(Ve^{-tf})$ which is

$$\Delta_0(Ve^{-tf}) = -tV \cdot \Delta_0(f)e^{-tf} + t^2 H(x,z,\nabla f)Ve^{-tf}.$$

But the Hamilton-Jacobi equation (6.3) yields

$$\begin{aligned}
\Delta_0(Ve^{-tf}) =& - tV \cdot \Delta_0(f)e^{-tf} + t^2\Big(f - \sum_{m=1}^3 \tau_m \frac{\partial f}{\partial \tau_m}\Big)Ve^{-tf} \\
=& - te^{-tf}\Big(V\Delta_0(f) + \sum_{m=1}^3 \frac{\partial}{\partial \tau_m}(\tau_m V)\Big) + t^2 e^{-tf} Vf\Big) \\
&+ t\sum_{m=1}^3 \frac{\partial}{\partial \tau_m}\big(\tau_m e^{-tf} V\big).
\end{aligned}$$

We also know that

$$\int_0^K t^{2n+3} e^{-tf} f dt = -K^{2n+3} e^{-Kf} + (2n+3) \int_0^K t^{2n+2} e^{-tf} dt.$$

This implies that

$$\begin{aligned}
\int_{\mathbb{R}^3} d\tau \int_0^K t^{2n+1} e^{-tf} t^2 f V dt =& - K^{2n+3} \int_{\mathbb{R}^3} Ve^{-Kf} d\tau \\
&+ (2n+3) \int_0^K t^{2n+2} dt \int_{\mathbb{R}^3} e^{-tf} V d\tau.
\end{aligned} \tag{7.24}$$

Hence,

$$\begin{aligned}
\Delta_0 \tilde{G}_K(x,z) =& \frac{-1}{\Gamma(2n+2)} \int_0^K t^{2n+2} dt \int_{\mathbb{R}^3} e^{-tf} \Big[\sum_{m=1}^3 \frac{\partial(\tau_m V)}{\partial \tau_m} + (\Delta_0 f - 2n - 3)V\Big] d\tau \\
&- \frac{K^{2n+3}}{\Gamma(2n+2)} \int_{\mathbb{R}^3} Ve^{-Kf} d\tau,
\end{aligned}$$

where we used (7.22) and (7.24). The first integral vanishes since V satisfies the generalized transport equation (7.7). Therefore, for $x \neq 0$,

$$\Delta_0 \tilde{G}_K(x,z) = -\frac{K^{2n+3}}{\Gamma(2n+2)} \int_{\mathbb{R}^3} V(\tau) e^{-Kf(x,z,\tau)} d\tau.$$

However, we may also change contour in the above integral and obtain

$$\Delta_0 \tilde{G}_K(x,z) = -\frac{K^{2n+3}}{\Gamma(2n+2)} \int_{\mathbb{R}^3} V(\tau + i\varepsilon\tilde{z}) e^{-Kf(x,z,\tau+i\varepsilon\tilde{z})} d\tau, \quad \text{for} \quad x \neq 0. \tag{7.25}$$

Since $\tilde{G}_K(x,z)$ is smooth everywhere, the integral (7.25) provides the value of $\Delta_0 \tilde{G}_K(x,z)$ at every point where the integral is convergent. It follows that $\Delta_0 \tilde{G}_K(x,z)$ is equal to

$$-\frac{K^{2n+3}}{(2n+1)!} \int_{\mathbb{R}^3} V(\tau + i\varepsilon\tilde{z}) e^{-Kf(x,z,\tau+i\varepsilon\tilde{z})} d\tau$$

everywhere and to $-\frac{K^{2n+3}}{(2n+1)!} \int_{\mathbb{R}^3} V(\tau) e^{-Kf(x,z,\tau)} d\tau$ almost everywhere. Furthermore, for (x,z) in a compact set U with U disjoint from the origin,

$$\mathrm{Re}(f)(x,z,\tau + i\varepsilon\tilde{z}) \geq \kappa > 0.$$

Hence $\Delta_0 \tilde{G}_K(x,z) \to 0$ uniformly as $K \to \infty$ on compact subsets of $\mathbb{R}^{4n} \times \mathbb{R}^3$ which is disjoint from the origin. Now we need to compute the L^1-norm of $\Delta_0 \tilde{G}_K(x,z)$. Since the integral $\int_{\mathbb{R}^3} V(\tau) e^{-tf(x,z,\tau)} d\tau$ coincides with the integral of the right-hand side of (7.25) almost everywhere, we can just compute the following

$$\mathcal{I} := -\frac{K^{2n+3}}{(2n+1)!} \Big| \int_{\mathbb{R}^{4n}} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} V(\tau) e^{-K[\gamma(x,\tau) - i\sum_{m=1}^3 \tau_m z_m]} d\tau dz dx \Big|.$$

Since the integral converges absolutely, we may interchange the order of the integration by Fubini's Theorem. Let us integrate the z-variable first.

$$\begin{aligned}
&\int_{\mathbb{R}^3} e^{-K[\gamma(x,\tau) - i\sum_{m=1}^3 \tau_m z_m]} dz = \int_{\mathbb{R}^3} e^{-\frac{K}{4}\sum_{l=1}^n |x|_l^2 |\tau|_l \coth(|\tau|_l) + iK\sum_{m=1}^3 \tau_m z_m]} dz \\
=& e^{-K[1/4\sum_{l=1}^n |x|_l^2 |\tau|_l \coth(|\tau|_l)} \int_{\mathbb{R}^3} e^{iK\sum_{m=1}^3 \tau_m z_m} dz \\
=& -e^{-K[1/4\sum_{l=1}^n |x|_l^2 |\tau|_l \coth(|\tau|_l)} K^{-3} (2\pi)^3 \mathcal{F}^{-1}(1) = K^{-3}(2\pi)^3 \delta(\tau).
\end{aligned}$$

Here $\mathcal{F}(g)(\xi) = \int_{\mathbb{R}^3} g(x) e^{-2\pi i x\cdot\xi} dx$ is the Fourier transform of the function $g(x)$. We know that

$$\lim_{\tau \to 0} \gamma(x,\tau) = \lim_{\tau \to 0} \frac{1}{4} \sum_{l=1}^n |x|_l^2 |\tau|_l \coth(|\tau|_l) = \frac{1}{4}|x|^2$$

and

$$\lim_{\tau \to 0} V(\tau) = \lim_{\tau \to 0} \prod_{l=1}^n \frac{|\tau|_l^2}{\sinh^2 |\tau|_l} = 1.$$

It follows that

$$K^{-3}(2\pi)^3 \int_{\mathbb{R}^3} V(\tau) e^{-K\gamma(x,\tau)} \delta(\tau) d\tau = \frac{(2\pi)^3}{K^3} e^{-\frac{K}{4}|x|^2}$$

in the sense of distribution. Finally, one has

$$\frac{1}{K^3}(2\pi)^3 \int_{\mathbb{R}^{4n}} e^{-\frac{K}{4}|x|^2} dx = \frac{(2\pi)^3}{K^3} 2^{4n} \frac{\pi^{2n}}{K^{2n}}.$$

This gives us

$$\mathcal{I} := -\frac{K^{2n+3}}{(2n+1)!}\frac{(2\pi)^3}{K^3}2^{4n}\frac{\pi^{2n}}{K^{2n}} = -2^{2n}\frac{(2\pi)^{2n+3}}{(2n+1)!}.$$

This proves that $\Delta_0\tilde{G}_K(x,z) \to 0$ uniformly on compact sets on $\mathbb{R}^{4n}\times\mathbb{R}^3$ disjoint from the origin with a constant integral over $\mathbb{R}^{4n}\times\mathbb{R}^3$. This means that when $K\to\infty$,

$$\Delta_0\tilde{G}_K(x,z) \;\to\; -2^{2n}\frac{(2\pi)^{2n+3}}{(2n+1)!}\delta_{(0,0)}.$$

On the other hand, $\tilde{G}_K(x,z)\to\tilde{G}(x,z)$ in $L^1_{loc}(\mathbb{R}^{4n+3})$ as $K\to\infty$. Hence,

$$\Delta_0\tilde{G}_K(x,z) \;\to\; \Delta_0\tilde{G}(x,z)$$

in the sense of distribution. Therefore,

$$\Delta_0\tilde{G}(x,z) = -\frac{2^{2n}(2\pi)^{2n+3}}{(2n+1)!}\delta_{(0,0)}.$$

The proof of the theorem is therefore complete. □

The symmetry of homogeneous ℍ-type groups allows us to deduce another form of Green's function related to the homogeneous norm (see, for instance [10]).

8. Estimates of the fundamental solution

In this section, we discuss sharp estimates for the integral operator induced by the fundamental solution $G(x,z)$:

$$\mathbf{G}(g)(x,z) = G*g(x,z) = \int_{Q^n} G(y,w)g((y,w)^{-1}\cdot(x,z))dydw$$

in $L^p_k(Q^n)$ Sobolev spaces, Hardy-Sobolev spaces $H^p_k(Q^n)$ for $k\in\mathbf{Z}_+$, $0<p<\infty$ and Lipschitz spaces Λ_β, Γ_β with $\beta>0$. We consider here the sub-Laplacian Δ_0. It is easy to see from the group law that the operator is homogeneous of degree -2 under that non-isotropic dilation:

$$\delta_\lambda:\,(x,z)\;\to\;(\lambda x,\lambda^2 z).$$

Hence, in general, the homogeneous degree of Q^n is $4n+6$. As in Folland-Stein [14] and Koranyi [20], we may define a homogeneous norm

$$|(x,z)| = \Big(\big(\sum_{j=1}^n\sum_{k=1}^4 x_{jk}^2\big)^2 + \sum_{j=1}^3 z_j^2\Big)^{\frac14}.$$

Then we may define a pseudo metric by $\rho(\mathbf{x},\mathbf{y}) = |\mathbf{x}\cdot\mathbf{y}^{-1}|$ where $\mathbf{x}=(x,z)$ and $\mathbf{y}=(y,w)$. Then it is easy to see that ρ is equivalent to the Carnot-Carathéodory metric. Using this metric, one may obtain estimates of $G(x,z)$ in various function spaces. From the discussion in Section 7, we know that the fundamental solution $G(x,z)$ with singularity at the origin is a homogeneous kernel of degree $-4n-4$. In fact, the fundamental solution $G(x,z)$ satisfies the following size estimates:

$$\big|G(x,z)\big| \le \frac{C_1|(x,z)|^2}{\big|B((x,z),|(x,z)|)\big|},\qquad \Big|\frac{\partial^{|\alpha|}G(x,z)}{\partial x_{11}^{\alpha_{11}}\cdots\partial x_{4n}^{\alpha_{4n}}}\Big| \le \frac{C_2|(x,z)|^{2-|\alpha|}}{\big|B((x,z),|(x,z)|)\big|},$$

and

$$\Big|\frac{\partial^{\beta_1+\beta_2+\beta_3}G(x,z)}{\partial z_1^{\beta_1}\partial z_2^{\beta_2}\partial z_3^{\beta_3}}\Big| \le \frac{C_3|(x,z)|^{2-2(\beta_1+\beta_2+\beta_3)}}{\big|B((x,z),|(x,z)|)\big|}.$$

Therefore, it is a locally integrable function. From classical results, it is easy to see that the operator $\mathbf{G}$ originally defined on the Schwartz space $\mathcal{S}(Q^n)$ can be extended to a bounded operator from $L^p_{loc}(Q^n)$ into $L^p_{loc}(Q^n)$ for $1 \le p \le \infty$ (see [4] and [14]). Moreover, $\mathbf{G}$ is a smoothing operator. Hence, we may allow to differentiate the kernel $G(x, z)$. Moreover, for any $l = 1, \dots, n$ and $k = 1, \dots, 4$, $X_{kl}\mathbf{G}$ originally defined on $\mathcal{S}(Q^n)$ can be extended to a bounded operator from $L^p_{loc}(Q^n)$ into $L^p_{loc}(Q^n)$ for $1 \le p \le \infty$. The problem now reduces to looking at the second derivatives (in the horizontal directions) of $G(x, z)$. Before we go further, let us recall some basic definitions and properties of several functions spaces.

• **Lipschitz spaces** As in [14], we define the space $\Gamma_\beta(Q^n)$ as the set of all bounded functions g with compact support on Q^n such that

$$\sup_{(x,z),(y,w)\in Q^n} \Big| g((x,z)\cdot(y,w)^{-1}) - g(x,z) \Big| \le C \cdot |(y,w)|^\beta, \qquad 0 < \beta < 1.$$

When $k < \beta < k+1$ with $k = 1, 2, ...$, we may define $g \in \Gamma_\beta(Q^n)$ as the set of all C^k functions g with compact support on Q^n such that $\mathcal{P}(X, X')g \in \Gamma_\beta(Q^n)$, where $\mathcal{P}(X, X')$ is a monomial of degree k in vector fields X, $X' \in V_1$. For integral value k, one may define the space $\Gamma_\beta(Q^n)$ by interpolation. Furthermore, since $|(x, z)| \le A\|(x, z)\|^{1/2}$ for $|(x, z)|$ small, one may conclude that $\Gamma_\beta(Q^n) \subset \Lambda_{\beta/2}(Q^n)$. Here $\|(x, z)\|$ is the Euclidean distance between the point (x, z) and the origin and $\Lambda_\alpha(Q^n)$ is the isotropic Lipschitz space on Q^n. As usual, the space $\Lambda_\alpha(Q^n)$ is defined as the collection of all bounded functions g with compact support on Q^n such that

$$\sup_{(x,z),(y,w)\in Q^n} \Big\| g((x,z)\cdot(y,w)^{-1}) - g(x,z) \Big\| \le C \cdot \|(y,w)\|^\alpha, \quad 0 < \alpha < 1.$$

For $k < \alpha < k+1$ with $k = 1, 2, ...$, we may define $g \in \Lambda_\alpha(Q^n)$ is the set of all C^k functions g with compact support on Q^n such that $\nabla^k g \in \Lambda_\alpha(Q^n)$.

• **Sobolev spaces.** One may define the non-isotropic Sobolev spaces $S^p_k(Q^n)$ with $k \in \mathbf{Z}_+$ and $1 < p < \infty$ as follows

$$S^p_k(Q^n) = \big\{ f : Q^n \to \mathbb{C} : \ f \in L^p(Q^n),\ \mathcal{P}(X, X')f \in L^p(Q^n) \big\},$$

where $\mathcal{P}(X, X')$ is a monomial of degree k in vector fields X, $X' \in V_1$. Here $L^p(Q^n)$ is the L^p Lebesgue space.

• **Hardy spaces.** The Hardy space $H^p(Q^n)$ with $0 < p < \infty$ originally defined by maximal function as follows: a distribution f defined on Q^n belongs $H^p(Q^n)$ if and only if the maximal function

$$\mathcal{M}(f)(x, z) = \sup_{\varepsilon>0} \big| f * \phi_\varepsilon \big|(x, z) \ \in \ L^p(Q^n).$$

Here $\phi \in \mathcal{S}(Q^n)$ with $\int_{Q^n} \phi(x, z)dxdz = 1$ and polyradial (see Chapter 4 in Folland-Stein [15]). As usual,

$$\phi_\varepsilon(x, z) = \varepsilon^{-(4n+6)}\phi\big(\varepsilon^{-1}x, \varepsilon^{-2}z\big).$$

The space $H^p(Q^n)$ can be defined by atomic decomposition and maximal functions.

Definition 8.1. A $H^p(Q^n)$ p-atom ($0 < p \le 1$) is a compactly supported function $a(x, z)$ such that the following conditions hold:

(1) (*size condition*): there is a Q^n-ball $B_{\mathbf{x}_0} = B_{(x_0,z_0)}(r) = \{\mathbf{x} = (x, z) \in Q^n : \rho(\mathbf{x}_0, \mathbf{x}) \le r\}$ whose closure contains $\mathrm{supp}(a)$ such that $\|a(\mathbf{x})\|_{L^\infty} \le |B_{\mathbf{x}_0}|^{-1/p}$;

(2) (*moment condition*):

$$\int_{Q^n} a(x, z)\mathcal{P}(x, z)\, dxdz = 0$$

for all monomials $\mathcal{P}(\mathbf{x}) = \mathcal{P}(x,z)$ such that $\mathcal{P}(\mathbf{x}) = \Big(\prod_{j=1}^n x_{j1}^{\alpha_{j1}} \cdots x_{j4}^{\alpha_{j4}}\Big) z_1^{\beta_1} z_2^{\beta_2} z_3^{\beta_3}$ and

$$\sum_{k=1}^{4}\sum_{j=1}^{n} \alpha_{jk} + 2(\beta_1+\beta_2+\beta_3) \le \Big[(4n+6)\big(\frac{1}{p}-1\big)\Big].$$

Here $[s]$ is the integral part of s.

Using the idea of atomic decomposition, we give the definition of $H^p(Q^n)$ as follows

$$H^p(Q^n) = \Big\{ f \in \mathcal{S}'(Q^n) : \ f = \sum_{k=1}^{\infty} \lambda_k a_k, \ \text{where } a_k \text{ are } p\text{-atoms}, \ \sum_{k=1}^{n} |\lambda_k|^p < \infty \Big\},$$

and we define

$$\|f\|^p_{H^p(Q^n)} = \inf \Big\{ \sum_{k=1}^{n} |\lambda_k|^p \Big\},$$

where the infimum is taken over all possible atomic decompositions of f. Then "norm" $\|f\|^p_{H^p(Q^n)}$ is comparable to the ℓ^p norm of the sequence $\{\lambda_k\}$ and the $L^p(Q^n)$ norm of the maximal function $\mathcal{M}(f)$.

Now, following notations in [4], for $g \in \mathcal{S}(Q^n)$, one has

$$\begin{aligned} X_{jk}X_{j'k'}\mathbf{G}(g)(x,z) = \lim_{\varepsilon\to 0^+} & \int_{\rho(y,w)=\varepsilon} (X_{jk}G)(y,w)(X_{j'k'}g)((x,z)\cdot(y,w)^{-1})\,dydw \\ & - \lim_{\varepsilon\to 0^+} \int_{\rho(y,w)\ge\varepsilon} (X_{jk}X_{j'k'}G)(y,w) g((x,z)\cdot(y,w)^{-1})\,dydw. \end{aligned}$$

The second term above is a generalized Calderón-Zygmund operator in the sense of [14] and Koranyi-Vagi [21] because $X_{jk}X_{j'k'}G$ is a kernel homogeneous of degree 0 satisfying mean value zero property. An operator $\mathbf{K}$ is said to be a generalized Calderón-Zygmund operator if the following two conditions are satisfied:
(i) $\mathbf{K}$ can be extended as a bounded operator on $L^2(Q^n)$,
(ii) There is a sequence of positive constant numbers $\{C_j\}$ such that for each $j \in \mathbb{N}$,
(8.1)

$$\left(\int_{2^j\rho(\mathbf{x}_2,\mathbf{x}_3)\le\rho(\mathbf{x}_1,\mathbf{x}_2)<2^{j+1}\rho(\mathbf{x}_2,\mathbf{x}_3)} |K(\mathbf{x}_1,\mathbf{x}_2)-K(\mathbf{x}_1,\mathbf{x}_3)|^q\,d\mathbf{x}_1\right)^{\frac{1}{q}} \le C_j\cdot\big(2^j\rho(\mathbf{x}_2,\mathbf{x}_3)\big)^{-\frac{4n+6}{q'}}$$

and
(8.2)

$$\left(\int_{2^j\rho(\mathbf{x}_2,\mathbf{x}_3)\le\rho(\mathbf{x}_1,\mathbf{x}_2)<2^{j+1}\rho(\mathbf{x}_2,\mathbf{x}_3)} |K(\mathbf{x}_2,\mathbf{x}_1)-K(\mathbf{x}_3,\mathbf{x}_1)|^q\,d\mathbf{x}_1\right)^{\frac{1}{q}} \le C_j\cdot\big(2^j\rho(\mathbf{x}_2,\mathbf{x}_3)\big)^{-\frac{4n+6}{q'}},$$

here (q,q') is a fixed pair of positive numbers with $\frac{1}{q}+\frac{1}{q'}=1$ and $1<q'<2$. We need the following theorem to complete our discussion on the estimates for $\mathbf{G}$. The proof can be found in [14] and [15].

Theorem 8.1. *Let $\mathbf{K}$ be a generalized Calderón-Zygmund operator. Assume that the kernel satisfies conditions (8.1)-(8.2) with $\{C_j\}\in\ell^1$. Then*

$$\|\mathbf{K}(f)\|_{L^p(Q^n)} \le C_p\|f\|_{L^p(Q^n)}, \quad 1<p<\infty;$$

and

$$\Big|\{\mathbf{x}\in Q^n : \ |\mathbf{K}(f)(\mathbf{x})|>\lambda\}\Big| \le \frac{C}{\lambda}\|f\|_{L^1(Q^n)}, \quad \lambda>0.$$

When $0 < p \leq 1$, we may consider the boundedness of generalized Calderón-Zygmund operator acting on Hardy spaces $H^p(Q^n)$. We have the following theorem (see [11]).

Theorem 8.2. *Let $\mathbf{K}$ be a generalized Calderón-Zygmund operator. Suppose the following two conditions hold:*
(i) *(kernel assumption): there exist* $s \geq \big[(4n+6)(\frac{1}{p}-1)\big]$ *and* $\varepsilon > \frac{1}{p}-1$ *such that*

$$\{2^{(4n+6)j(2\gamma-1)}(C_j)^2\} \in \ell^1 \quad \textit{with} \quad \gamma = 1 - \frac{1}{p} + \varepsilon,$$

(ii) *(adjoint operator assumption):* $\mathbf{K}^*(x_{1l}^{\alpha_{1l}} \cdots x_{4l}^{\alpha_{4l}} z_1^{\beta_1} z_2^{\beta_2} z_3^{\beta_3}) = 0$ *with*

$$\sum_{k=1}^{4}\sum_{l=1}^{n} \alpha_{kl} + \sum_{m=1}^{3} \beta_i \leq \big[(4n+6)(\frac{1}{p}-1)\big],$$

where $\mathbf{K}^*$ *is the adjoint operator of* $\mathbf{K}$.
Then $\mathbf{K}$ *can be extended as a bounded operator from* $H^p(Q^n)$ *into* $H^p(Q^n)$ *for* $0 < p \leq 1$.

We first make the following observation. For any polynomial $\mathcal{P}(X)$ of degree k in the horizontal vector fields $X_{1l}, \ldots, X_{4l}$ with $l = 1, \ldots, n$, there exists another polynomial $\tilde{\mathcal{P}}(X)$ such that the following identity holds:

$$\mathcal{P}(X)\mathbf{G} = \mathbf{G}\tilde{\mathcal{P}}(X). \tag{8.3}$$

Now we may apply the above theorems and (8.3) to conclude the following result.

Theorem 8.3. *The fundamental solution* $G(x, z)$ *for the operator* Δ_0 *defines an operator* $\mathbf{G}$ *which satisfies the following sharp estimates:*

i) $XX'\mathbf{G}$ *defines a bounded operator from the Sobolev space* $S_k^p(Q^n)$ *into Sobolev space* $S_k^p(Q^n)$ *for* $k \in \mathbf{Z}_+$, $1 < p < \infty$ *and for all* X, $X' \in V_1$*;*

ii) $Z\mathbf{G}$ *defines a bounded operator from the non-isotropic Sobolev space* $S_k^p(Q^n)$ *into* $S_k^p(Q^n)$ *for* $k \in \mathbf{Z}_+$, $1 < p < \infty$ *and for all* $Z \in V_2$*;*

iii) $XX'\mathbf{G}$ *defines a bounded operator from* $H_k^p(Q^n)$ *into* $H_k^p(Q^n)$ *for* $k \in \mathbf{Z}_+$ *and* $0 < p \leq 1$ *and for all* X, $X' \in V_1$*;*

iv) $Z\mathbf{G}$ *defines a bounded operator from* $H_k^p(Q^n)$ *into* $H_k^p(Q^n)$ *for* $k \in \mathbf{Z}_+$ *and* $0 < p \leq 1$ *and for all* $Z \in V_2$*;*

v) $\mathbf{G}$ *defines a bounded operator from* $\Lambda_\beta(Q^n)$ *into* $\Gamma_{\beta+2}(Q^n) \cap \Lambda_{\beta+1}(Q^n)$ *for* $0 < \beta < \infty$.

In conclusion, $\mathbf{G}$ *gains two in horizontal directions and only gains one in missing directions.*

References

[1] Auslander L., MacKenzie R. E. *Introduction to differentiable manifolds.* McGraw - Hill Book Co., Inc., New York-Toronto-London 1963, 219 pp.

[2] Baez, J.C., *The Octonions*, Bull. Amer. Math. soc. **39** (2001), no. 2, 145–205.

[3] Balogh Z., Manfredi J., and Tyson J. *Fundamental solution for the Q-Laplacian and sharp Moser-Trudinger inequality in Carnot groups.* J. Funct. Anal. **204** (2003), no. 1, 35–49.

[4] Beals R., Gaveau B., and Greiner P. C. *The Green function of model step two hypoelliptic operators and the analysis of certain tangential Cauchy Riemann Complexes.* Adv. Math. **121** (1996), 288–345.

[5] Beals R., Gaveau B., and Greiner P. C. *Complex Hamiltonian mechanics and parametrices for subelliptic Laplacians, I, II, III.* Bull. Sci. Math., **21** (1997), no. 1–3, 1–36, 97–149, 195–259.

[6] Beals R., Gaveau B., and Greiner P. C. *Hamilton-Jacobi theory and the heat kernel on Heisenberg groups.* J. Math. Pures Appl. **79** (2000), no. 7, 633–689.

[7] Calin O., Chang D. C., and Greiner P. C. *On a step* $2(k+1)$ *sub-Riemannian manifold.* J. Geom. Anal., **14** (2004), no. 1, 1–18

[8] Calin O., Chang D. C., and Greiner P. C. *Geometric Analysis on the Heisenberg Group and Its Generalizations*, to be published in AMS/IP series in advanced mathematics, International Press, Cambridge, Massachusetts, 2005.

[9] Capogna L. and Garofalo N. *Regularity of minimizers of the calculus of variations in Carnot groups via hypoellipticity of systems of Hörmander type.* J. Eur. Math. Soc. (JEMS) **5** (2003), no. 1, 1–40.

[10] Capogna L., Danielli D., and Garofalo N. *Capacitary estimates and the local behavior of solutions of nonlinear subelliptic equations.* Amer. J. Math. **118** (1996), no. 6, 1153–1196.

[11] Chang D. C. *Estimates for singular integral operators with mixed type homogeneities in Hardy classes*, Math. Ann., **287** (1990), 303-322.

[12] Chang D. C. and Markina I. *Geometric analysis on quaternion* $\mathbb{H}$*-type groups*, J. Geom. Anal., (2006).

[13] Chow W. L. *Wei-Liang Über Systeme von linearen partiellen Differentialgleichungen erster Ordnung.* Math. Ann. **117** (1939), 98–105.

[14] Folland G. B. and Stein E. M. *Estimates for the* $\bar{\partial}_b$ *complex and analysis on the Heisenberg group*, Comm. Pure Appl. Math. **27** (1974), 429–522.

[15] Folland G. B. and Stein E. M. *Hardy Spaces on Homogeneous Groups*, Math. Notes **28**, Princeton University Press, Princeton, N.J., 1982.

[16] Gaveau B. *Principe de moindre action, propagation de la chaleur et estimées sous elliptiques sur certains groupes nilpotents*, Acta Math. **139** (1977), no. 1-2, 95–153.

[17] Hörmander L. *Hypoelliptic second order differential equations.* Acta Math. **119** (1967) 147–171.

[18] Kaplan A. *Fundamental solutions for a class of hypoelliptic PDE generated by composition of quadratics forms.* Trans. Amer. Math. Soc. **258** (1980), no. 1, 147–153.

[19] Kaplan A. *On the geometry of groups of Heisenberg type.* Bull. London Math. Soc. **15** (1983), no. 1, 35–42.

[20] Korányi A. *Geometric properties of Heisenberg-type groups.* Adv. in Math. **56** (1985), no. 1, 28–38.

[21] Korányi A. and Vagi S. *Singular integrals in homogeneous spaces and some problems of classical analysis.* Ann. Scuola Norm. Sup. Pisa **25** (1971), 575–648.

[22] Montgomery R., Shapiro M., and Stolin A. *A nonintegrable sub-Riemannian geodesic flow on a Carnot group.* J. Dynam. Control Systems **3** (1997), no. 4, 519–530.

[23] Reimann H. M. *Rigidity of* $\mathbb{H}$*-type groups.* Math. Z. **237** (2001), no. 4, 697–725.

[24] Strichartz R. S. *Sub-Riemannian geometry.* J. Differential Geom. **24** (1986), no. 2, 221–263; Correction, ibid. **30** (1989), 595-596.

Department of Mathematics, Georgetown University, Washington D.C. 20057, USA
E-mail address: `chang@georgetown.edu`

Department of Mathematics, University of Bergen, Johannes Brunsgate 12, Bergen 5008, Norway
E-mail address: `irina.markina@uib.no`